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A PROCEDURE FOR ORDERING OBJECT PAIRS CONSISTENT WITH THE MULTIDIMENSIONAL UNFOLDING MODEL

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A procedure for ordering object (stimulus) pairs based on individual preference ratings is described. The basic assumption is that individual responses are consistent with a nonmetric multidimensional unfolding model. The method requires data where a numerical response is independently generated for each individual-object pair. In conjunction with a nonmetric multidimensional scaling procedure, it provides a vehicle for recovering meaningful object configurations.

Key words: nonmetric, scaling, measurement, line-of-sight.

Considerable attention has been devoted to the multidimensional unfolding problem [Coombs, 1950, 1964; Hays & Bennett, 1961; Kruskal, 1964a; Schoneman, 1970; Davidson, 1972, 1973; Zinnes & Griggs, 1974]. However, little effort has been expended developing measures of inter-object similarity consistent with the unfolding model apart from those implicit in these general solutions. Given the known data sensitivity of the multidimensional unfolding methods, this neglect is somewhat surprising. In the absence of a satisfactory similarity measure, investigators have turned to theoretically inappropriate alternatives such as Pearson product-moment correlations and sums of squared differences between pairs of objects to assess inter-object similarity [Rabinowitz, Note 1; Jones, 1974].

In this paper the line-of-sight method for ordering the pairwise similarity of a set of objects will be described. The method is nonmetric and suitable for large populations of individuals. In combination with standard nonmetric multidimensional scaling procedures that can be used to scale the object points [Kruskal, 1964a, 1964b; Guttman, 1968; Young, 1968], it provides a vehicle for recovering meaningful object configurations. In addition, if individual points are subsequently scaled in the fixed object space, the line-of-sight method offers the potential for a more robust solution to the nonmetric unfolding problem [Kruskal & Carroll, 1969].

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Two assumptions are made in deriving the measure which are *not* integral to the classic nonmetric multidimensional unfolding problem. One is a distribution assumption: it is assumed that individual points are widely dispersed in the object space. This assumption will be made more explicit later in the paper. The other involves the monotonic functions which relate spatial position to preferential response: it is assumed initially that all subjects share the same monotonic function. Later this assumption is relaxed to allow for a variety of functions; however, a unique function for every subject is not allowed. Since rank data implies a separate monotone function for each individual, it cannot be appropriately analyzed with this method. This procedure requires data where a numerical response is independently generated for each individual-object pair which reflects the utility (degree of preference) of the object for the individual.

In the discussion which will follow, the underlying logic of the approach will be illustrated using a Euclidean metric. It will be shown that this logic can be generalized to any Minkowski metric under any monotonic distortion. A method will then be proposed which is consistent with this formal development. Finally, the method will be applied to some artificially generated and some real data and the results observed.

Heuristic Discussion

Suppose we are interested in finding the distance between points A and B in Figure 1. If we pick any third point (for instance Y) on the line determined by A and B and between the two points, and add the distances from each of the points to the third point, the resulting sum would equal the distance of A to B. This we will denote as d_{AB}. Thus, $d_{AY} + d_{BY} = d_{AB}$. If we pick any point which lies off the line segment, such as Y', the sum of the distances would be *greater than* the distance between A and B. Similarly, if we choose X lying on the line determined by A and B but outside the A to B segment and calculate the absolute difference between the distance between X and A and the distance between X and B, that absolute difference would equal the distance between A and B. Thus, $|d_{AX} - d_{BX}| = d_{AB}$. Notice that if we pick a point not on the line, such as X', the difference would be *less than* the distance between A and B.

If we treat X and Y as individual ideal points and A and B as object points, and if responses to the objects are directly proportional to Euclidean distance, the *minimum sum* of responses over all subjects to a pair of objects would form an upper bound on the distance between the two objects and the *maximum absolute difference* of responses across all subjects would form a lower bound. Since we are not interested in assuming that responses are proportional to distance, we will not pursue this theme. More crucial to our purpose is the observation that, if some individual points are sufficiently close to the line, both on the segment between the two points and on the

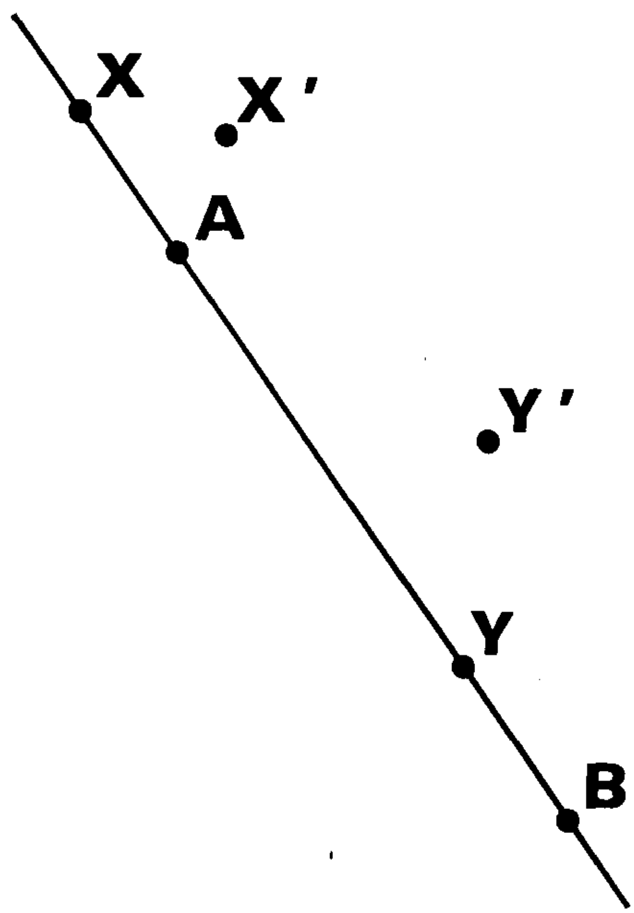

FIGURE 1
Illustration of some rudimentary properties of Euclidean spaces.

segment outside the two points, it might be possible to estimate the relative distance between the two points using *either* the maximum absolute difference or the minimum sum.

Formal Discussion

Our initial observations were made on the basis of a Euclidean metric space; several features are general to any metric space.

First, in any metric space, the minimum sum that can be obtained by adding the distances from any third point to each of two fixed points is the actual distance between the fixed points:

$$d_{AB} \leq d_{AX} + d_{BX} .$$

This is the triangle inequality. Second, in any metric space, the maximum absolute difference that can be obtained by subtracting the distances of any third point from each of two fixed points is the actual distance between the fixed points:

$$d_{AB} \geq |d_{AX} - d_{BX}| .$$

To show this, assume the contrary; assume that $|d_{AX} - d_{BX}| > d_{AB}$ for some point X. Without loss of generality, let $d_{AX} > d_{BX}$; then $d_{AX} - d_{BX} > d_{AB}$, whence $d_{AX} > d_{AB} + d_{BX}$ which violates the triangle inequality. Hence the contrary assumption is false.

In a Euclidean space, points lying on a line determined by two fixed points can be used to determine the length of the line segment between the points; this is also true in any normed real vector space, which includes Minkowski metric spaces. (For a general discussion of real vector spaces see Royden [1968].)

Lemma. In any normed real vector space, given two fixed points A and B, $A \neq B$, and any point Y on the line segment between A and B, $d_{AY} + d_{BY} = d_{AB}$. Given any point Y lying on the line determined by A and B and outside the segment between A and B, $|d_{AY} - d_{BY}| = d_{AB}$.

Proof. Part 1: d_{AB} in a normed vector space is defined as $||A - B||$ where $||$ denotes the norm. The line segment between A and B is the set of points $\{tA + (1 - t)B : 0 \leq t \leq 1\}$. Let Y, an arbitrary point on that segment, be written as $t_0A + (1 - t_0)B$ or equivalently, $t_0(A - B) + B$. Then

$$d_{AY} = ||A - [t_0(A - B) + B]|| = ||(1 - t_0)(A - B)|| = |1 - t_0| \, ||A - B|| = (1 - t_0) \, d_{AB} ;$$

$$d_{BY} = ||B - [t_0(A - B) + B]|| = ||-t_0(A - B)|| = |-t_0| \, ||A - B|| = t_0 \, d_{AB} .$$

Hence, $d_{AY} + d_{BY} = (1 - t_0)d_{AB} + t_0 d_{AB} = d_{AB}$.

Part 2: Let Y be an arbitrary point on the line determined by A and B but outside the segment; without loss of generality, assume $d_{AY} \geq d_{BY}$. Using Part 1, $d_{AB} + d_{BY} = d_{AY}$. Hence, $d_{AB} = d_{AY} - d_{BY}$. □

We are now in a position to consider the measurement problem. Let a finite set of q object points in an arbitrary normed real vector space be denoted as $C = \{C_1 , C_2 , \cdots , C_q\}$ and a finite set of p individual ideal points as $V = \{V_1 , V_2 , \cdots , V_p\}$. The model we initially consider is

$$r_{V_i C_j} = f(d_{V_i C_j}),$$

where $r_{V_i C_j}$ is the preferential response of individual i to object j, $d_{V_i C_j}$ is the distance of the point representing individual i from the point representing object j, and f is any monotonic function from the positive reals to a finite set of reals. (Since the direction of f has no actual bearing on the results, we shall henceforth assume f is increasing.)

The measure of interobject similarity will depend on finding maximum absolute differences and minimum sums of preference scores for each pair of objects over all individuals. The one major assumption which must be made beyond the appropriateness of the model involves the distribution of individual points. We now offer three definitions to make this assumption explicit, and then state the line-of-sight theorem. The first term to be defined is *relevant region*. Since f has only a finite range, there exists a distance DMAX such that all distances greater than DMAX have, as their associated value under f, the maximum value in the range of the function, FMAX. Individual points which are further from any object point than DMAX are irrelevant in the subsequent measurements since the absolute difference for each pair of objects would be zero and the sum for each pair of objects would be twice FMAX.

Definition 1: Relevant Region. The relevant region associated with C and f is the set of points P which satisfy the condition $f(d_{PC_i}) < FMAX$ *for some* C_i *in C.*

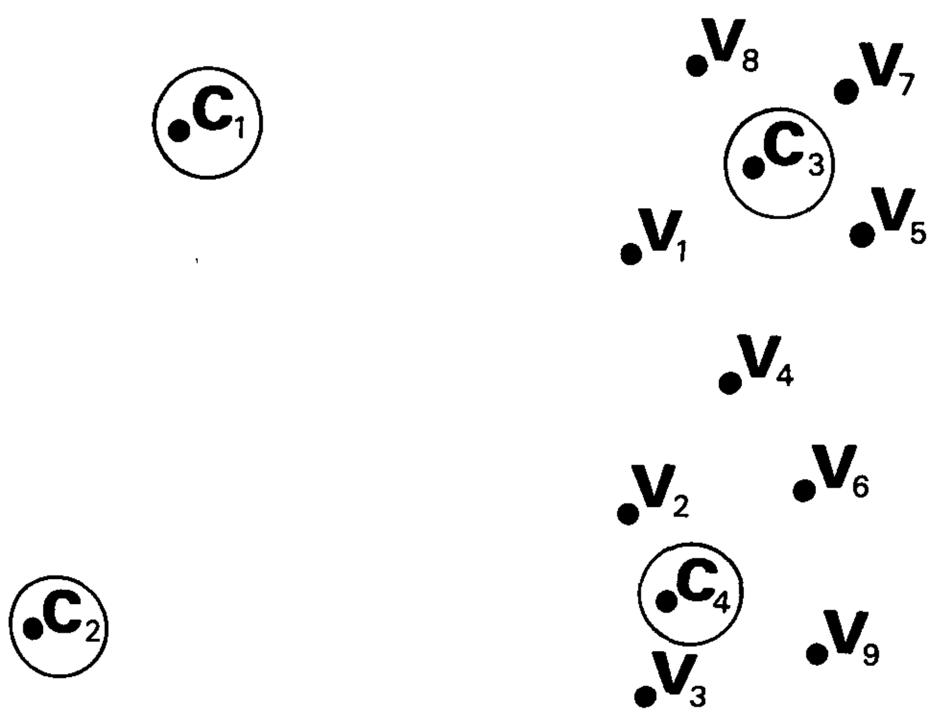

FIGURE 2
No individual points (V's) in the area of C_1 and C_2; maximum absolute difference and minimum sum are both poor indicators of interobject distance.

The next term to be defined is *well filled*. If the maximum differences and minimum sums associated with pairs of points in C are to accurately order the interpoint distances, it seems reasonable that the set of individual points should be widely distributed in the space. For example, if all the individual points were distributed in a cluster, as in Figure 2, a rank order based on minimum sums would locate those points away from the cluster, such as C_1 and C_2, farther apart than the actual distance would warrant. On the other hand, a rank order of pairs based on maximum difference would likely place them too close together. However, if there were even a few points in that empty region around C_1 and C_2, as in Figure 3, the chances of making a reasonable ranking using minimum sums or maximum differences would be considerably enhanced, though not insured.

To insure an accurate ranking based on minimum sums or maximum differences, two factors must be considered. The first relates to the similarity of the distances between pairs of points in the object set C. The more similar a pair of distances are, the more difficult it is to order them properly; hence,

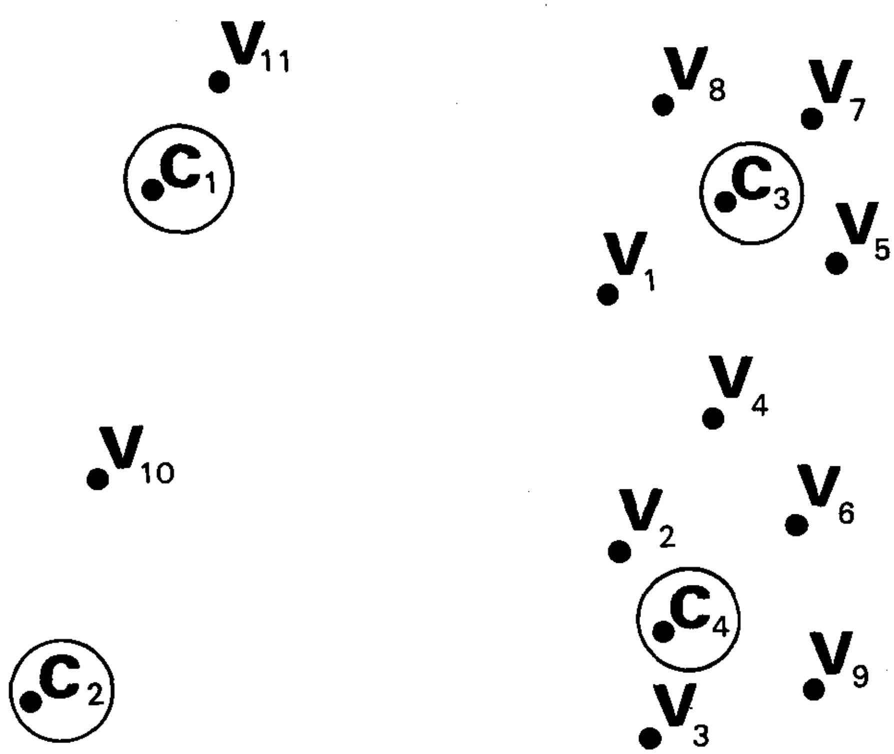

FIGURE 3
Two individual points (V's) in the area of C_1 and C_2; maximum absolute difference and minimum sum both are better indicators of interobject distance.

the more densely the individual points must be distributed in the space. The second relates to the behavior of the function f. If the function tends to have as its values very low scores for those distances close to zero and if it increases very rapidly for scores greater than zero, the idiosyncratic appearance of a point in V quite close to a point in C could radically alter the minimum sums and maximum differences associated with that point in C.

Definition 2: Well Filled. A set of points P is well filled by V with respect to C and f, if for every point in P there exists a point V_0 *in V, such that* $d_{V_0 P} < e/2$, *where e is the lesser of* e^* *and* e'; e^* *is the minimum* $|d_{C_i C_j} - d_{C_k C_l}|$ *over all quadruples in C, such that* $i \neq j$, $k \neq l$, *and the pair i, j is not identical to pair k, l; and* e' *is the largest value such that* $f(e' - \Delta) = FMIN$, *where FMIN is the minimum value in the range of f and* Δ *is any real number such that* $0 < \Delta \leq e'$.

The last term to be defined is *critical region.* The proof of the line-of-sight theorem will depend only on the density of individual points in the hypercylinders surrounding the lines determined by pairs of points in the object set.

Definition 3: Critical Region. The critical region A associated with C and f is the intersection of the set of points which compose the relevant region and the set of points lying with e/2 units of the lines determined by the points in C, where e is defined as in the definition of well filled above.

We now state the theorem.

Theorem. If A, the critical region, is well filled by V, and if $d_{C_i C_j} > d_{C_k C_l}$, there exists a V^* and $V^{\#}$ in V such that

$$f(d_{V_x C_i}) + f(d_{V_x C_j}) \geq f(d_{V^* C_k}) + f(d_{V^* C_l}), \quad (1)$$

and

$$|f(d_{V^{\#} C_i}) - f(d_{V^{\#} C_j})| \geq |f(d_{V_0 C_k}) - f(d_{V_0 C_l})|, \quad (2)$$

where V_x minimizes the left term of (1) and V_0 maximizes the right term of (2) over all points in V.

Figure 4 provides a geometric referent to accompany the proof.

Proof. Part 1: Assume that V_x has been found, and then locate P^* $d_{V_x C_i} - e/2$ units from C_k in the direction of C_l. (However, if $d_{V_x C_i} - e/2 > d_{C_k C_l}$, locate P^* at C_l, or if $d_{V_x C_i} < e/2$, locate P^* at C_k.) Select V^* within $e/2$ units of P^*. By virtue of the triangular inequality and the lemma, this construction strategy insures that $d_{V^* C_k} < d_{V_x C_i}$ and $d_{V^* C_l} < d_{V_x C_j}$. (Or in cases where the first inequality does not hold, $d_{V^* C_k} < e$; and in cases where the second inequality does not hold, $d_{V^* C_l} < e$.) Since f is monotonic (or since $f(<e) = FMIN$), $f(d_{V^* C_k}) \leq f(d_{V_x C_i})$ and $f(d_{V^* C_l}) \leq f(d_{V_x C_j})$. Therefore, $f(d_{V^* C_k}) + f(d_{V^* C_l}) \leq f(d_{V_x C_i}) + f(d_{V_x C_j})$.

Part 2: Assume that V_0 has been found and with no loss of generality let $d_{V_0 C_k} \geq d_{V_0 C_l}$. Locate $P^{\#}$ $d_{V_0 C_k} + e/2$ units from C_i in the direction of C_j. (However, if $d_{V_0 C_k} + e/2 < d_{C_i C_j}$, locate $P^{\#}$ at C_j.) Select $V^{\#}$ within $e/2$ units of $P^{\#}$. By virtue of the triangular inequality, Observation 2, and the

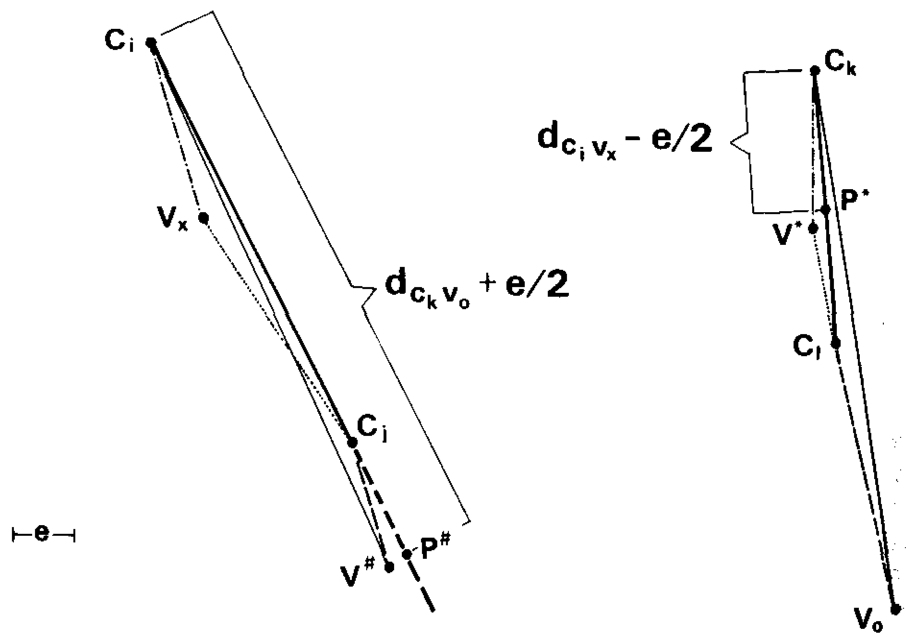

FIGURE 4
Illustration to accompany the proof of the line-of-sight theorem.

lemma, this construction strategy insures that $d_{V^\# C_i} > d_{V_o C_k}$ and $d_{V^\# C_j} < d_{V_o C_l}$. (Or in cases where $d_{V^\# C_j} \geq d_{V_o C_l}$, that $d_{V^\# C_j} < e$.) Since f is monotonic (or since $f(<e) = FMIN$), $f(d_{V^\# C_i}) \geq f(d_{V_o C_i})$ and $f(d_{V^\# C_j}) \leq f(d_{V_o C_l})$. Therefore, $f(d_{V^\# C_i}) - f(d_{V^\# C_j}) \geq f(d_{V_o C_k}) - f(d_{V_o C_l})$.

Finally, the assumption of a generally shared monotone function can be relaxed. The analytic model now becomes $r_{V_{ki} C_j} = f_k(d_{V_{ki} C_j})$, where $r_{V_{ki} C_j}$ is the response of individual i in subset k to object j, and f_k is the monotone function shared by members of the subset. To apply this more general model, it is necessary to assume that each A_k , the critical region with respect to f_k and C, is well filled by its corresponding V_k . The result then follows directly from the theorem and is stated in the following corollary.

Corollary. Let $F = \{f_1 , f_2 , \cdots , f_s\}$ be a set of s increasing monotone functions from the positive reals into a finite set of reals. Let

$$C = \{C_1 , C_2 , \cdots , C_q\}$$

and

$$V = \{V_{11} , V_{12} , \cdots , V_{1p_1} , V_{21} , \cdots , V_{2p_2} , \cdots , V_{sp_s}\}$$

be sets of points in an arbitrary normed real vector space. Let $F(d_{V_{ki} C_j})$ be defined as $f_k(d_{V_{ki} C_j})$; let A_k be the critical region with respect to f_k and C; and let $V_k = \{V_{k1} , V_{k2} , \cdots , V_{kp_k}\}$.

If each A_k is well filled by its corresponding V_k, and if $d_{C_iC_j} > d_{C_kC_l}$, there exists a V^* and $V^\#$ in V such that

$$F(d_{V_xC_i}) + F(d_{V_xC_j}) \geq F(d_{V^*C_k}) + F(d_{V^*C_l}), \tag{3}$$

and

$$|F(d_{V^\# C_i}) - F(d_{V^\# C_j})| \geq |F(d_{V_0C_k}) - F(d_{V_0C_l})|, \tag{4}$$

where V_x minimizes the left term of (3) and V_0 maximizes the right term of (4) over all points in V.

Procedure

With the addition of this corollary, the formal development is completed. If the assumptions underlying the method are satisfied, a ranking of pairs of objects based either on the minimum sums or maximum absolute differences will correctly order the object pairs. However, two problems are likely to arise when the method is applied. First, a ranking based on either minimum sums or maximum differences will usually not discriminate adequately between pairs, and only a weak partial ordering will be possible. For example, if subjects are rating objects on a nine-point scale which runs from "like very much" to "dislike very much," the sums would have a maximum 17 point range and the differences a maximum range of nine points, while (with as few as ten objects) there are 45 pairs to be ordered. Second, if a researcher expects that the subjects are generally responding in a fashion consistent with the model but error is present, then basing a measure on a single extreme response for each pair can be undesirable. To ameliorate both problems, more information than the single smallest sum or single largest difference must be used. The procedure we shall propose is one of several which could be devised. When developing it we were most interested in the case where the number of individuals is quite large ($p > 500$) and considerable error is present in the data. While the procedure lacks a rigorous foundation, it is consistent with the formal development and does seem to work empirically.

For notational convenience, we shall define several matrices at this time; upper case letters will be used for the matrices and their corresponding lower case letters for the matrix entries. The subscript i will be reserved for individuals, j and k for objects, and m for object pairs; i will run from 1 to p, where p is the number of individuals; j and k will run from 1 to q, where q is the number of objects; and m will run from 1 to $q(q-1)/2$. In addition, the subscripts h and g will be used and will run from 1 to p, but will not directly correspond to individuals. To help clarify any ambiguities, several of the matrices are illustrated in Table 1.

1) R is the p individual by q object matrix in which the subject preference responses are recorded. Hence, r_{ij} is the response of individual i to object j. (In keeping with the formal development, we will assume that low values

TABLE 1

Illustration of Matrices Used in the Procedure

R
Basic Rating†

	Object 1	Object 2	Object 3
Individual 1	20	40	20
Individual 2	70	50	60
Individual 3	70	30	60
Individual 4	10	60	60
Individual 5	30	20	20

	S* Unsorted sum			D* Unsorted absolute difference		
	Object Pair 1,2	Object Pair 1,3	Object Pair 2,3	Object Pair 1,2	Object Pair 1,3	Object Pair 2,3
Individual 1	60	40	60	20	0	20
Individual 2	120	130	110	20	10	10
Individual 3	100	130	90	40	10	30
Individual 4	70	70	120	50	50	0
Individual 5	50	50	40	10	10	0

	S Sorted sum			D Sorted absolute difference		
	Object Pair 1,2	Object Pair 1,3	Object Pair 2,3	Object Pair 1,2	Object Pair 1,3	Object Pair 2,3
Level 1	50	40	40	50	50	30
Level 2	60	50	60	40	10	20
Level 3	70	70	90	20	10	10
Level 4	100	130	110	20	10	0
Level 5	120	130	120	10	0	0

	B Combined (B = S + D)			$\hat{B}$ Cumulated B		
	Object Pair 1,2	Object Pair 1,3	Object Pair 2,3	Object Pair 1,2	Object Pair 1,3	Object Pair 2,3
Level 1	100	90	70	100	90	70
Level 2	100	60	80	200	150	150
Level 3	90	80	100	290	230	250
Level 4	120	140	110	410	370	360
Level 5	130	130	120	540	500	480

† The lower the number the more favorably the object is evaluated.

are associated with favorable ratings and high values with unfavorable ratings. If high values are associated with favorable ratings, each rating could be subtracted from an arbitrary constant and made consistent with this assumption.)

2) S^* is the p individual by $q(q-1)/2$ object pair matrix in which the sum of responses to the two objects are recorded. Thus $s_{im}^* = r_{ij} + r_{ik}$.
3) D^* is the p individual by $q(q-1)/2$ object pair matrix in which the absolute difference of responses to the two objects are recorded. Thus $d_{im}^* = |r_{ij} - r_{ik}|$.
4) S is the p by $q(q-1)/2$ matrix derived from S^* by sorting within *each column* of S^* from smallest to largest. Hence, while there are p rows in the S matrix, they *do not* correspond to individuals, but rather to what we shall call *levels.*
5) D is a p by $q(q-1)/2$ matrix which is derived from D^* by sorting within each column of D^* from largest to smallest.
6) B is the matrix sum of S and D; thus $B = S + D$.
7) $\hat{B}$ is the cumulative B matrix; thus $\hat{b}_{gm} = \sum_{h=1}^{g} b_{hm}$ (Note: The *order* of values in the gth row of $\hat{B}$ is equivalent to the *order* which would be obtained were one to calculate the mean of the first g rows of B.)
8) $\hat{S}$ is the cumulative S matrix; thus $\hat{s}_{gm} = \sum_{h=1}^{g} s_{hm}$
9) $\hat{D}$ is the cumulative D matrix; thus $\hat{d}_{gm} = \sum_{h=1}^{g} d_{hm}$

The first row of S and the first row of D correspond to the single smallest sum and the single largest difference for each pair; a partial order based on the values in both these rows would be consistent with the pairwise ordering (if the assumptions are satisfied). The sum of the two comprise the first row of B, which would similarly be order preserving. The first row of B would also provide at least as much discrimination as either the smallest sum row or the largest difference row. Each successive row of S and D and hence B will be increasingly *inappropriate* as a device for sorting the pairs, because of the density assumptions implicit in using information beyond the smallest sum and the largest difference. However, if the number of subjects is large and the distribution of individual points is similar to (tends to overlap) the distribution of object points, the increase in distortion should not be severe.

A natural way to use information beyond the first row of B is to average several of the first few rows and use the mean as the basis for the pairwise ranking. Such an averaging procedure is particularly appropriate if random error might induce unreliability in the initial ranking. Two criteria seem reasonable to use as a guide in deciding how many rows to include in the average. First, unless the number of rows used is small relative to the number of individuals, the method would be inconsistent with the formal development. Second, good discrimination between pairs is essential, if subsequent scaling is to be meaningful. A simple computer-oriented procedure was devised with these criteria in mind.

For each row of $\hat{B}$, an ordering of the $q(q - 1)/2$ pairs is made. With respect to each ordering, two values are calculated; one measures the extent of pairwise discrimination, the other the extent to which the density criterion is being strained. They are defined as follows:

$$\text{DISCRIM} = \frac{\text{Number of distinct ranking positions} - 1}{\text{Total number of pairs} - 1};$$

$$\text{DENSE} = \frac{\text{Number of subjects} - \text{row of } \hat{B} \text{ used to form the ranking}}{\text{Number of subjects} - 1}.$$

These are combined into a single "adequacy" measure, ADEQU = $\text{DISCRIM} \cdot \text{DENSE}^{\text{EXPO}}$ where EXPO = 3. The exponent is included because without it the expression tends to be determined almost exclusively by the degree of pairwise discrimination. The rows of $\hat{B}$ are then sequentially examined and when the measure does not increase after a fixed number of rows are examined (either four or $p/150$, whichever is larger), the row of $\hat{B}$ associated with the largest adequacy value is used for the ranking.

Experience with the method indicates that if the row selected is beyond the $p/25$th row, the resulting order is suspect and the procedure should be repeated with the value of the exponent raised. Since the ranking is based on the cumulative matrix, the pairwise order tends to be quite stable from row to row, and the exact row selected is not particularly important. Indeed, an alternate procedure of selecting the single row of $\hat{B}$ with the best pairwise discrimination in the first $p/40$ rows seems to produce virtually identical results. In general, if there are five or fewer response alternatives on the original rating scale, the data do not possess enough information to satisfactorily apply the procedure.

Goodness of Fit

The procedure for obtaining the pairwise ranking uses both the differences and sums together. One way to test how well the assumptions underlying the method are met is to compare the ranking obtained using just the differences to the ranking obtained using just the sums. With random data these two rankings tend to be uncorrelated. If the assumptions of the method are satisfied, the rankings should be virtually identical. Hence, the Spearman rho correlation between the row of $\hat{S}$ and the row of $\hat{D}$ which corresponds to the row of $\hat{B}$ used to form the pairwise ranking should provide a measure of goodness of fit.

If this fit is poor, either the method is faulty or the unfolding model is inappropriate. In general, if each object is very favorable evaluated by a reasonable number of individuals, the unfolding model's applicability should be questioned. However, if some objects receive no or extremely few very favorable evaluations, this method is not suitable and the appropriateness of the model cannot be assessed.

In no instance does the quality of fit provide an indication of the dimensionality of the space. The method is based directly on distances; hence, if the model is appropriate and the method's assumptions are satisfied, the fit should be good regardless of the dimensionality of the space. Dimensionality must be determined in the course of the subsequent scaling procedure.

Monte Carlo Simulation

To test the adequacy of the line-of-sight procedure and to compare it with Pearson product-moment correlations and sums of squared differences, a Monte Carlo simulation was performed. Euclidean distances between 1,000 individual and 12 object points in two dimensions were computed under the following three distribution conditions.

The Overlapping Distribution. Four approximately normally distributed variables were independently generated, two with an N of 12 and two with an N of 1,000, each with a mean zero and a standard deviation of one. These respectively formed the object and individual configurations.

The Shifted Distribution. The two configurations generated above were retained, but the mean of the first dimension of the individual configuration was shifted 1.5 units to the right by adding 1.5 to each value on that dimension.

The Stretched Distribution. The two configurations generated in the first step were retained, but the *first* dimension of the object configuration and the *second* dimension of the individual configuration were stretched by multiplying each value on those dimensions by two.

A schematic drawing of the three distributions appears in Figure 5. They provide a reasonably diverse set of density conditions on which to apply the methods.

In order to test the nonmetric sensitivity of the methods, three additional sets of values were created by calculating the natural logarithms of the

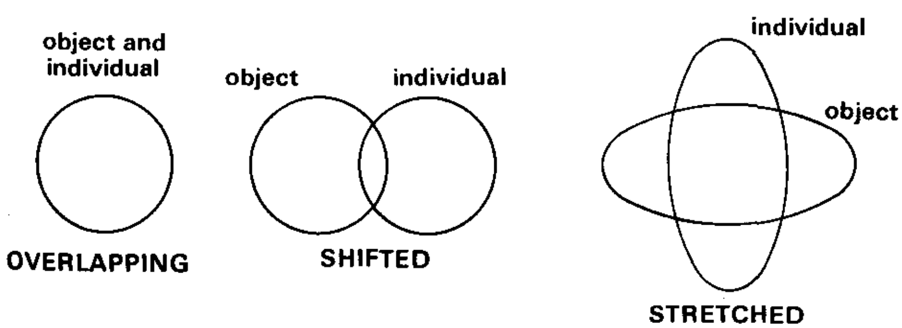

FIGURE 5
Schematic of three basic distributions used in the simulation. (The figures are drawn so that the probability of being within the boundary is approximately equal to 0.5).

distances. To assess the robustness of the methods to error, random normal error was added to the distances and transformed distances, creating six additional sets of values. In each instance the error term had a mean of zero and a standard deviation approximately equal to that of the set of distances or logged distances to which it was added. This left half the variance of these values unexplained and half directly dependent on the distances. Finally, each of the twelve sets of values were collapsed into nine-point scales by converting them to standard score units and coding them as follows:

$$
\begin{aligned}
X &< -1.75 = 1\\
-1.75 \leq X &< -1.25 = 2\\
-1.25 \leq X &< -0.75 = 3\\
-0.75 \leq X &< -0.25 = 4\\
-0.25 \leq X &< 0.25 = 5\\
0.25 \leq X &< 0.75 = 6\\
0.75 \leq X &< 1.25 = 7\\
1.25 \leq X &< 1.75 = 8\\
1.75 \leq X & \phantom{< -1.75} = 9.
\end{aligned}
$$

The three methods were then applied and the resulting similarity matrices scaled using a nonmetric multidimensional scaling routine. In Table 2, the Spearman rho correlation between the order of pairs recovered using each method and the underlying rank order is displayed, as is the squared Pearson product-moment correlation (the coefficient of metric determinacy [Young, 1970]) between the scaled and the underlying distances. Using the error-free data, the line-of-sight method is clearly superior in every instance. It performs least satisfactorily on the shifted distribution where the density assumptions are most strained.

The line-of-sight ordering and the ordering based on the Pearson product-moment correlations are both more sensitive to error than the sums of squared differences. The coefficient of metric determinacy remains virtually unchanged using the sums of squared differences, when the error term is added. Yet, even with the fairly large error component included in these data, the line-of-sight method uniformly provides a more data-consistent ordering than the sums of squared differences. In the two instances, overlapping + error and shifted + error, where the Spearman rho correlations produced using the line-of-sight method are most like those produced using sums of squared differences, the coefficient of metric determinacy is decidedly more favorable to the line-of-sight result. This is because the line-of-sight distortions were relatively

Summary of

	RH
Overlapping	.9
Shifted	.9
Stretched	.9
LN.Overlapping	.9
LN.Shifted	.9
LN.Stretched	.9
Overlapping + E	.9
Shifted + E	.8
Stretched + E	.9
LN.Overlapping + E	.9
LN.Shifted + E	.8
LN.Stretched + E	.9.

*RHO = Spearman r covered by the method point distances.

**Metric Deter. = the R^2 between the dis multidimensional scali tances between the poi

unsystematic, while the by the same systematic b

In Table 3 the line- addition, the Spearman r on the sums and the or The difference-sum corre tions in the recovery, and of fit.

In 1968, immediate individuals in a survey

TABLE 2

Summary of Results Using Simulated Data

	Line-of-Sight		Sums of Squared Differences		Pearson R	
	RHO*	Metric Deter.**	RHO*	Metric Deter.**	RHO*	Metric Deter.**
Overlapping	.998	.996	.975	.920	.931	.866
Shifted	.958	.957	.901	.714	.805	.693
Stretched	.982	.967	.861	.578	.771	.630
LN.Overlapping	.996	.995	.936	.804	.915	.835
LN.Shifted	.974	.962	.809	.593	.779	.665
LN.Stretched	.990	.988	.812	.409	.731	.550
Overlapping + E	.969	.958	.966	.920	.898	.787
Shifted + E	.888	.824	.881	.724	.620	.431
Stretched + E	.936	.868	.886	.608	.728	.599
LN.Overlapping + E	.977	.976	.922	.794	.903	.821
LN.Shifted + E	.881	.805	.791	.540	.813	.717
LN.Stretched + E	.954	.931	.806	.402	.706	.556

*RHO = Spearman rho calculated between the order of pairs recovered by the method and the order of pairs based on the actual interpoint distances.

**Metric Deter. = The coefficient of metric determinacy. This is the R^2 between the distances between the points following a nonmetric multidimensional scaling of the similarity matrix and the actual distances between the points.

unsystematic, while the sums of squared differences distortions were caused by the same systematic bias which influenced the error-free ranking.

In Table 3 the line-of-sight results are once again summarized, but in addition, the Spearman rho correlation between the order of pairs based only on the sums and the order of pairs based only on the differences appears. The difference-sum correlation does show a marked sensitivity to deteriorations in the recovery, and hence seems appropriate as an indicator of goodness of fit.

An Empirical Example

In 1968, immediately following the November presidential election, individuals in a survey sample of the United States were asked to respond

TABLE 3

Comparison of DIFF-SUM RHO to Actual Fit Measures

	DIFF-SUM RHO*	RHO**	METRIC DETER.***
Overlapping	.984	.998	.996
Shifted	.783	.958	.957
Stretched	.958	.982	.967
LN.Overlapping	.984	.996	.995
LN.Shifted	.717	.974	.962
LN.Stretched	.892	.990	.988
Overlapping + E	.885	.969	.958
Shifted + E	.433	.888	.824
Stretched + E	.845	.936	.868
LN.Overlapping + E	.825	.977	.976
LN.Shifted + E	.269	.881	.805
LN.Stretched + E	.704	.954	.931

*DIFF-SUM RHO = Spearman rho correlation between the rank order of pairs obtained using the larger differences with that obtained using the smaller sums.

**RHO = Spearman rho calculated between the order of pairs recovered by the method and the order of pairs based on the actual interpoint distances.

***METRIC DETER. = The coefficient of metric determinacy. This is the R^2 between the distances between the points following a nonmetric multidimensional scaling of the similarity matrix and the actual distances between the points.

to a set of political candidates on the basis of the subjective warmth they felt toward each of the candidates. Responses could vary from 100, which indicated extreme warmth, to zero, which indicated that the subject felt very cold toward the candidate. The cue for the response was the card which appears in Figure 6.

There is a considerable literature which suggests that an unfolding model might be correct to apply to these questions [Downs, 1957; Davis & Hinich, 1966; Davis, Hinich, & Ordeshook, 1970; Weisberg & Rusk, 1970; Mauser, 1972]. Hence, it is a reasonable data set on which to use the line-of-sight method. Traditional methods would be inappropriate for two reasons. First,

"FEELING" THERMOMETER

WARM	100°	___Very Warm or favorable feeling for candidate
	85°	___Good warm or favorable feeling for candidate
	70°	___Fairly warm or favorable
	60°	___A bit more warm or favorable than cold feeling
	50°	___No feeling at all for candidate
	40°	___A bit more cold or unfavorable feeling
	30°	___Fairly cold or unfavorable feeling
	15°	___Quite cold or unfavorable feeling
COLD	0°	___Very cold or unfavorable feeling

FIGURE 6
Illustration of "feeling" thermometer card.

the data set is large; it includes 1,136 subjects and 12 objects, which far exceeds the limitations of most of the unfolding programs. Second, and more critically, it is certain that the amount of error in the responses will be quite high [Stokes, 1963; Converse, 1964], even if the unfolding model is roughly

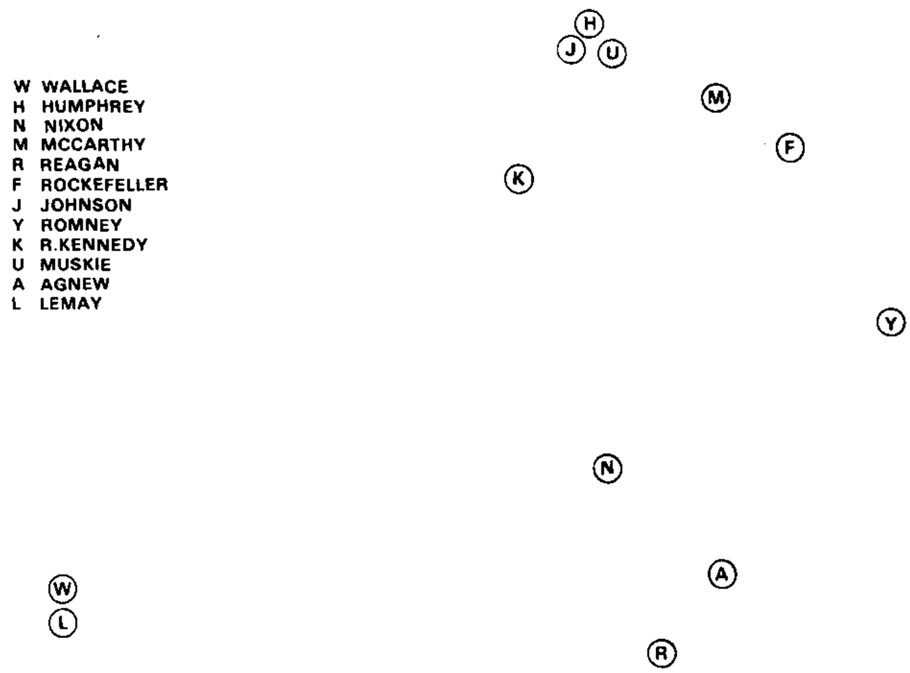

FIGURE 7
Configuration of candidate points from the 1968 U.S. presidential election campaign.

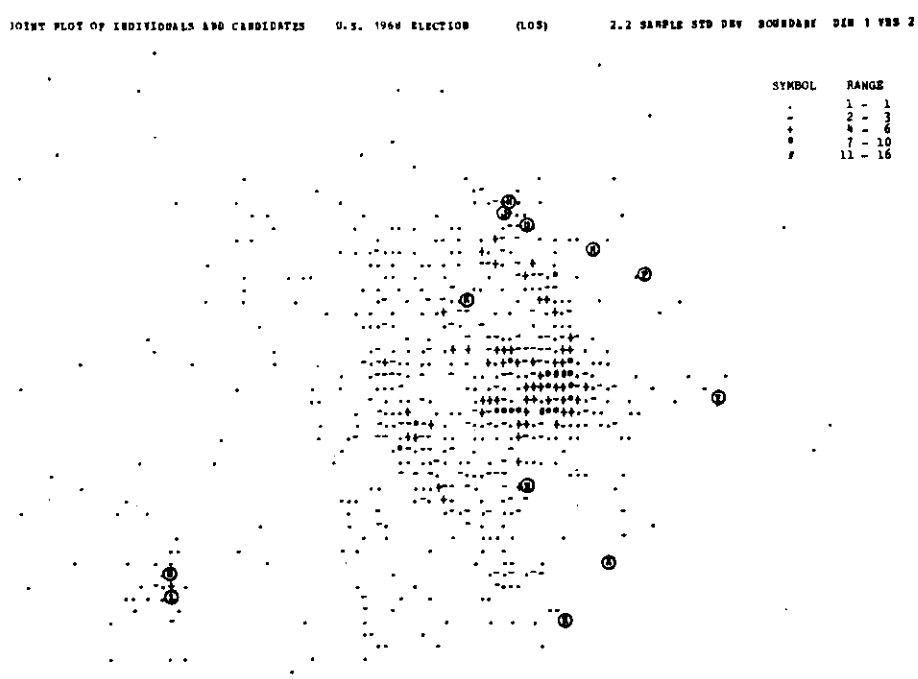

FIGURE 8
Joint plot of individuals and candidates.

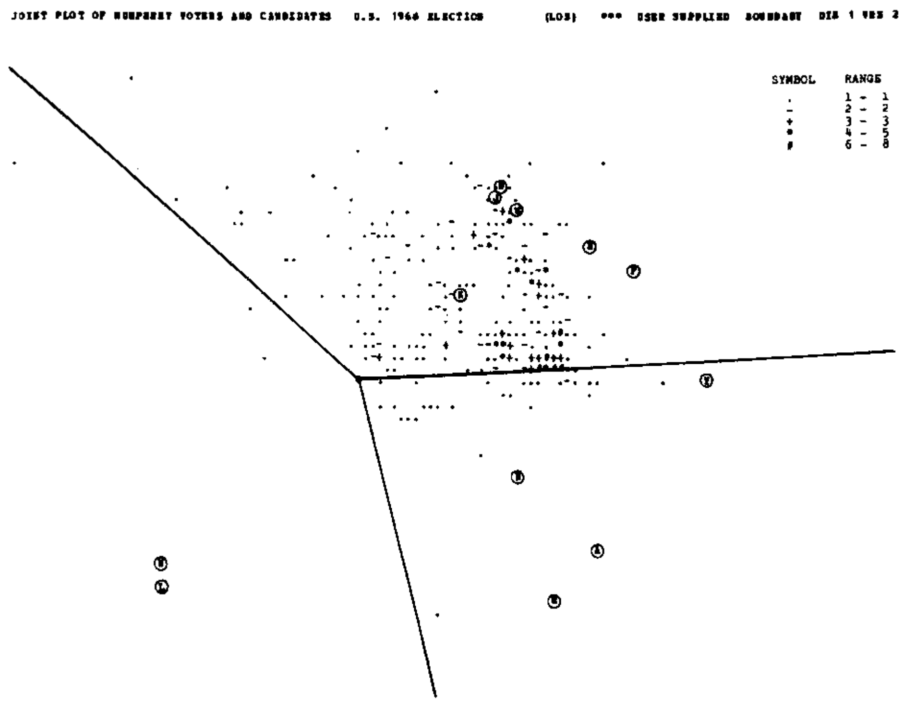

FIGURE 9
Joint plot of Humphrey voters and candidates.

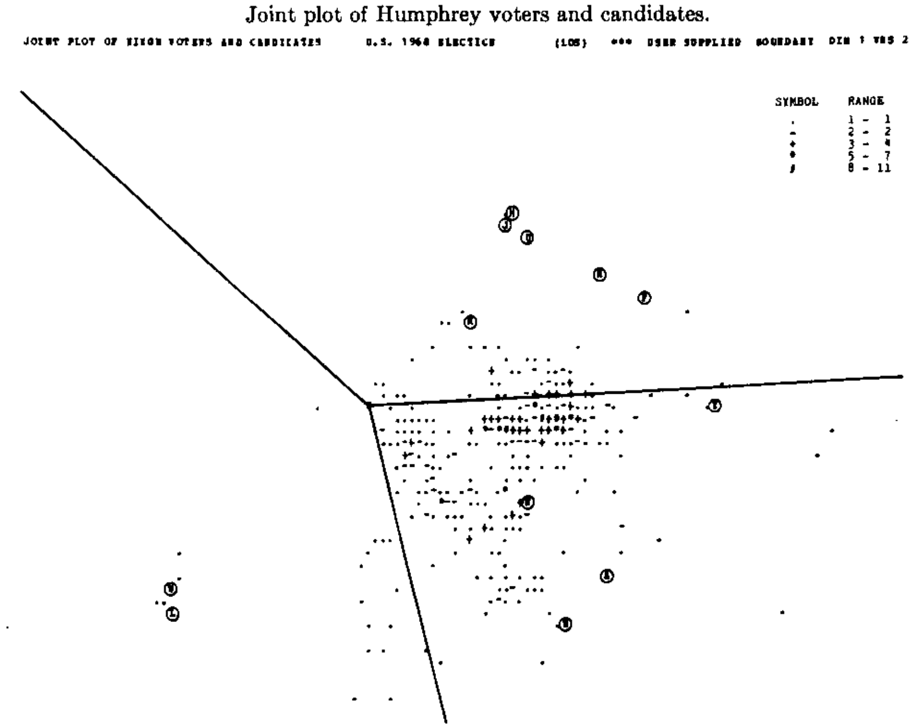

FIGURE 10
Joint plot of Nixon voters and candidates.

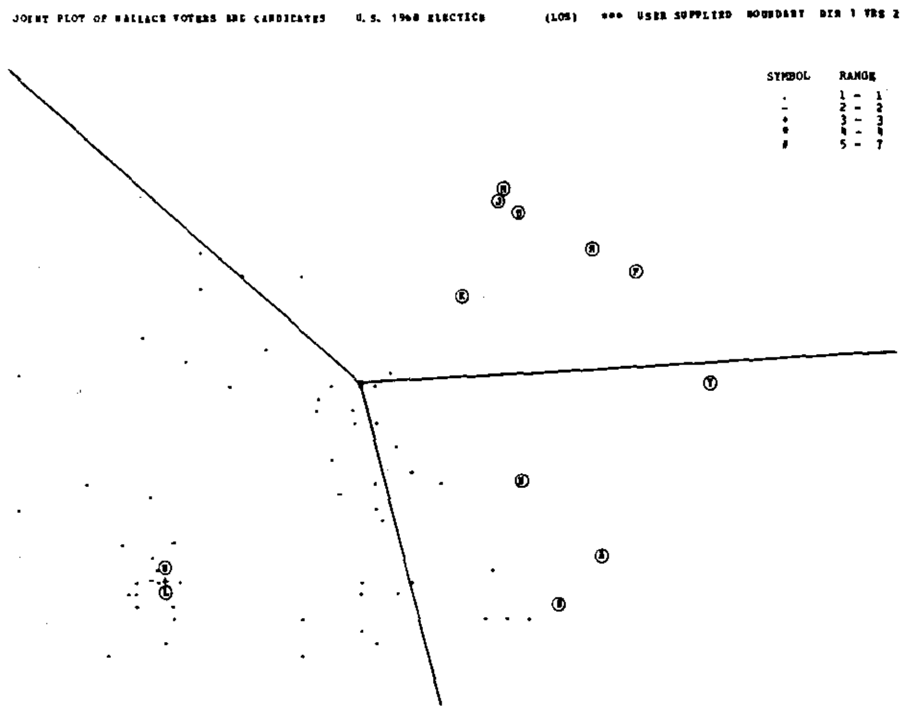

FIGURE 11
Joint plot of Wallace voters and candidates.

appropriate. This implies that traditional procedures would produce degenerate solutions [Kruskal & Carroll, 1969; Gleason, 1969].

Using the line-of-sight method, the Spearman rho correlation between the rank order of pairs based only on the absolute differences and that based only on the sums was 0.399. This low value makes sense if we anticipate considerable error and realize that well known candidates tend to draw more extreme responses. This salience effect produces larger maximum differences and smaller minimum sums for these candidates, thus decreasing the rho. The robustness shown by the line-of-sight measure with the artificial data suggests that subsequent scaling of the derived matrix is justified.

In Figure 7, the configuration recovered using a nonmetric multidimensional scaling routine on the line-of-sight matrix is displayed. Four basic groupings are present: a Democratic group including Humphrey, Johnson, and Muskie; a liberal group including McCarthy and Rockefeller; a Republican group including Nixon, Agnew, and Reagan; and an American Independent group including Wallace and Lemay. Neither Robert Kennedy nor Romney fall clearly into a single group. Kennedy, who had been assassinated by the time of the survey, is positioned between the Democratic and liberal

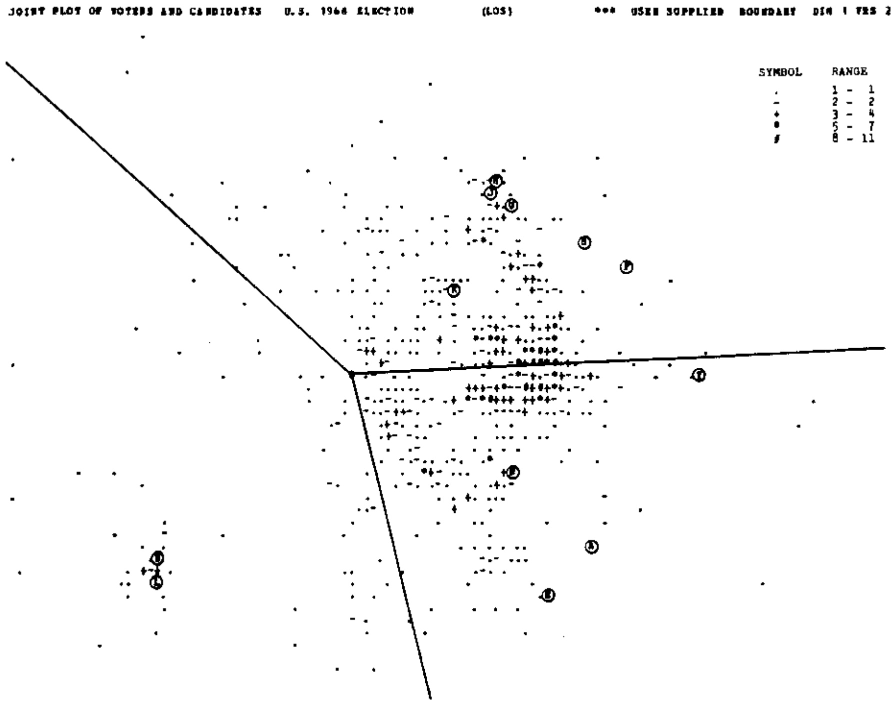

FIGURE 12
Joint plot of all voters and candidates.

groups and toward the center of the space. Romney, who faded from contention after conceding that he had been brainwashed about Vietnam, is located between the Republican and liberal group and away from the center of the space. The Democratic and Republican groups are antipodal as are the liberal and the American Independent groups. The two axes so defined are correlated with the liberal group closer to the Democratic and the American Independent closer to the Republican. This configuration fits very well with the general political analysis of the 1968 election (see for example, Converse, Miller, Rusk, Wolfe [1969]).

Yet before too much is made of the particular configuration, a natural question arises as to the substantive meaningfulness of the space. If the configuration recovered is an accurate reflection of voter perceptions of the candidates, it should be possible to locate points representing voters in the space. With this in mind, individual ideal points were externally scaled [Carroll, 1972] in the line-of-sight space. External scaling involves locating the individual points in the *fixed* object space. To do this a nonmetric multidimensional scaling program was developed. In this procedure, scaling for each individual point is separate from the scaling of any other point, and is based on an iterative negative gradient procedure which minimizes Kruskal's stress formula two [Kruskal, 1964a]. The joint space recovered appears in Figure 8.

The recovered space was divided by three lines into three regions: one associated with Nixon, one with Humphrey, and one with Wallace. Each of the dividing lines is restricted to run parallel to the actual bisector, but has been placed to allow for optimal prediction. Both the Humphrey-Nixon and Wallace-Nixon dividers are shifted away from the Nixon point. This is consistent with the generally observed phenomenon of increased appeal for newly-elected presidents [Mueller, 1970] and the post-election timing of the interview. Each individual was predicted to vote for the candidate in whose region his ideal point was located. On this basis 84.2 percent of the voters were predicted correctly.

In Figures 9, 10, and 11 plots of the Humphrey, Nixon, and Wallace voters appear. In Figure 12 the entire population of voters is plotted. In each of these plots the darkened lines indicate the respective region boundaries. An examination of the plots shows that the prediction is good, as most voters

TABLE 4

Demographic Group Summary

	Symbol	Correlation	Squared Correlation	% of Total Population in Group
Population	*	.982	.964	100.0
Black	B	.961	.924	9.2
White	W	.980	.960	89.5
White North (Non-South)	N	.980	.960	63.7
White South	S	.907	.822	25.9
White Deep South	D	.898	.806	3.8
Union Member in Family	U	.979	.958	25.0
Lower or Working Class	-	.976	.952	53.3
Upper or Middle Class	+	.979	.958	43.1
Low Income	¢	.980	.960	66.5
High Income	$	.965	.931	30.6
Grade School or Less	1	.965	.931	22.6
Some High School	2	.967	.935	17.9
Finished High School	3	.982	.964	32.0
Some College, Past High	4	.959	.920	27.2
Young (30-)	y	.973	.947	18.5
Middle Age	m	.979	.958	42.5
Older (50+)	o	.964	.929	38.6
Male	M	.971	.943	43.8
Female	F	.982	.964	56.2
Protestant (White)	P	.974	.949	62.0
Catholic (White)	C	.976	.953	20.2
Jewish (White)	J	.971	.943	2.7

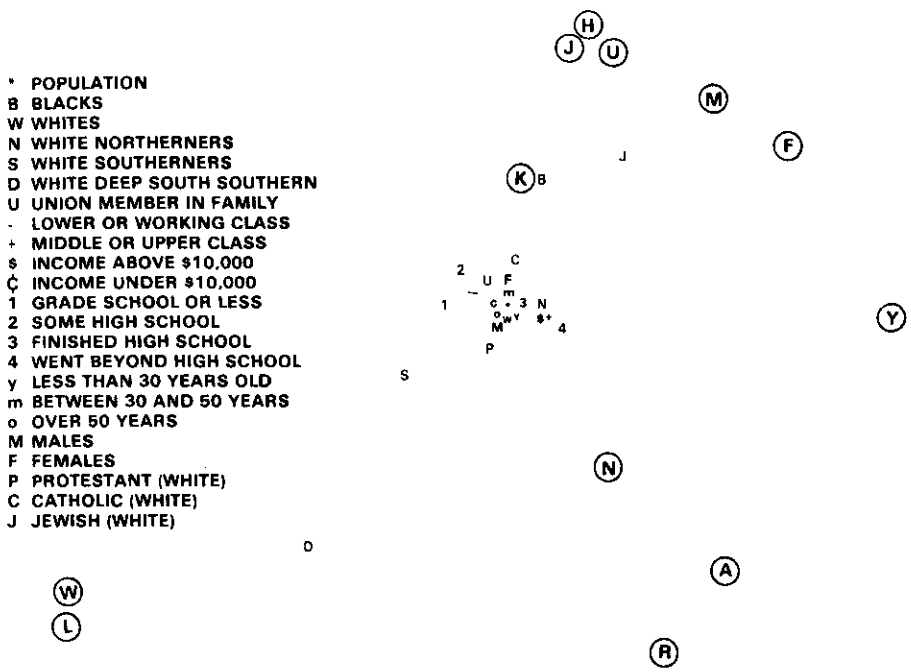

FIGURE 13
Joint plot of demographic groups and candidates.

do lie in the "correct" region. In addition, the incorrectly-predicted individuals tend to lie close to the relevant region boundary, which indicates that these individuals are not badly misplaced.

As a final test of the adequacy of this solution, the mean position of several demographic subgroups was calculated. The Pearson product-moment correlation between the distance to each of the candidates from the group point and the mean evaluation of the candidate by the members of the group was computed. The correlations and squared correlations appear in Table 4 and are generally high. The mean group positions are plotted in Figure 13. The locations are intuitively reasonable, and reflect the demographic differences noted by political researchers [Campbell, Converse, Miller, & Stokes, 1960].

Conclusion

The line-of-sight theory was developed and a method based on that theory was offered. The method performed well on the artificial data to which it was applied, and offers a more suitable approach to assessing interobject similarity than either Pearson product-moment correlations or sums of squared differences. Applied to the candidate choice responses, the method provided a first step in an external unfolding analysis which produced substantively meaningful results.

An IBM 360/Fortran IV computer program using this procedure is available upon request from the author.

REFERENCE NOTE

1. Rabinowitz, G. B. *Spatial models of electoral choice: An empirical analysis* (Working papers in methodology No. 7). Chapel Hill, North Carolina: Institute for Research in Social Science, 1973.

REFERENCES

Campbell, A., Converse, P. E., Miller, W. E., & Stokes, D. E. *The American Voter*. New York: Wiley, 1960.

Carroll, J. D. Individual differences and multidimensional scaling. In R. N. Shepard, A. K. Romney & S. B. Nerlove (Eds.), *Multi-dimensional scaling*. Volume 1, Theory. New York: Seminar Press, 1972.

Converse, P. E. The nature of belief systems in mass publics. In D. E. Apter (Ed.), *Ideology and discontent*. New York: Free Press, 1964.

Converse, P. E., Miller, W. E., Rusk, J. G., & Wolfe, A. C. Continuity and change in American politics: Parties and issues in the 1968 election. *American Political Science Review*, 1969, *63*, 1083–1105.

Coombs, C. H. Psychological scaling without a unit of measurement. *Psychological Review*, 1950, *57*, 145–158.

Coombs, C. H. *A theory of data*. New York: Wiley, 1964.

Davidson, J. A. A geometrical analysis of the unfolding model: non-degenerate solutions. *Psychometrika*, 1972, *37*, 193–216.

Davidson, J. A. A geometrical analysis of the unfolding model: general solutions. *Psychometrika*, 1973, *38*, 305–336.

Davis, O. A. & Hinich, M. A mathematical model of policy formation in a democratic society. In J. Bernd (Ed.), *Mathematical applications in political science II*. Dallas: Southern Methodist University Press, 1966.

Davis, O. A., Hinich M., & Ordeshook, P. An expository development of a mathematical model of the electoral process. *American Political Science Review*, 1970, *64*, 426–448.

Downs, A. *An economic theory of democracy*. New York: Harper and Row, 1957.

Gleason, T. *Multidimensional scaling of sociometric data*. Ann Arbor: Institute for Social Research, 1969.

Guttman, L. A general nonmetric technique for finding the smallest coordinate space for a configuration of points. *Psychometrika*, 1968, *33*, 469–506.

Hays, W. L. & Bennett, J. F. Multidimensional unfolding: determining configuration from complete rank order preference data. *Psychometrika*, 1961, *26*, 221–238.

Jones, B. D. Some considerations in the use of nonmetric multidimensional scaling. *Political Methodology*, 1974, *1*, 1–30.

Kruskal, J. B. Multidimensional scaling by optimizing goodness of fit to a nonmetric hypothesis. *Psychometrika*, 1964, *29*, 1–27(a).

Kruskal, J. B. Nonmetric multidimensional scaling: a numerical method. *Psychometrika*, 1964, *29*, 28–42(b).

Kruskal, J. B. & Carroll, J. D. Geometric models and badness of fit functions. In P. R. Krishnaiah (Ed.), *International symposium of multivariate analysis, Dayton, Ohio, 1968*. New York: Academic Press, 1969.

Mauser, G. A. A structural approach to predicting patterns of electoral substitution. In R. N. Shepard, A. K. Romney, and S. B. Nerlove (Eds.), *Multidimensional scaling.* Volume 2, Applications. New York: Seminar Press, 1972.

Mueller, J. E. Presidential popularity from Truman to Johnson. *American Political Science Review*, 1970, *66*, 979–995.

Royden, H. L. *Real analysis.* London: Macmillan, 1968.

Schoneman, P. H. On metric multidimensional unfolding. *Psychometrika*, 1970, *35*, 349–366.

Stokes, D. E. Spatial models of party competition. *American Political Science Review*, 1963, *57*, 368–377.

Weisberg, H. F. & Rusk, J. G. Dimensions of candidate evaluations. *American Political Science Review*, 1970, *64*, 1167–1185.

Young, F. W. TORSCA—A FORTRAN IV program for nonmetric multidimensional scaling. *Behavioral Science*, 1968, *13*, 343–344.

Young F. W. Nonmetric multidimensional scaling: Recovery of metric information. *Psychometrika*, 1970, *35*, 455–473.

Zinnes, J. L. & Griggs, R. A. Probabilistic, multidimensional unfolding analysis. *Psychometrika*, 1974, *39*, 327–350.

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