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MULTIDIMENSIONAL SCALING ❖337

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CONTENTS

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INTRODUCTION

Perhaps the most salient feature in the progress of multidimensional scaling (MDS) over the past 7 years since Cliff's (1973) chapter on "Scaling" has been the explosive growth in number and variety of models and methods, the proliferation of applications of MDS within many different fields, and a kind of semantic encroachment of the term MDS on other domains (e.g. factor analysis, test theory, analysis of variance or mathematical models). This semantic expansion of the term is not necessarily undesirable, since "multidimensional scaling," liberally speaking, could be taken to include much that has traditionally been identified with other areas of psychometrics or mathematical psychology. Broadly defined, multidimensional scaling comprises a family of geometric models for multidimensional representation of data and a corresponding set of methods for fitting such models to actual data. A much narrower definition would limit the term to spatial distance models for similarities, dissimilarities or other *proximities data.* The usage we espouse would include nonspatial (e.g. discrete geometric models such as tree structures) and nondistance (e.g. scalar product or projection) models that apply to nonproximities (e.g. preference or other "dominance") data as well as to proximities.

Because of this methodological and semantic expansion of the field, it seems to us that the major service a reviewer can do for readers is to attempt to put some order into what may appear as chaos: that is, to impose a taxonomy on the field. This task is our goal. At the outset, we state our disclaimers. Our taxonomy is only one of many possible ways of organizing the field; we view the classification as provisional, relevant to the field as it now is and not as it may be some years in the future. In effect, our taxonomy might be regarded as a subjectively derived meta-multidimensional scaling (and/or clustering) of the current state of multidimensional scaling. We hope that the taxonomy will facilitate readers' understanding of the work reviewed herein, as well as of the chapter itself.

A NEW TAXONOMY OF MEASUREMENT DATA AND OF MULTIDIMENSIONAL MEASUREMENT MODELS

The present taxonomy can be viewed as an attempt to update and generalize Coombs' (1964) *A Theory of Data,* although there are many ways in which our taxonomy departs significantly from Coombs', so that our approach is not, strictly speaking, a generalization. Still, the clearest antecedent is *A Theory of Data,* and Coombs (personal communication) has indicated that except for our use of "data" where he would use "observations," he finds no conflict between his (1964; also see Coombs 1979) taxonomy and the present one. Our viewpoint has also been influenced by Shepard's (1972b) taxonomy of data and of methods of analysis. Finally, the term "modes" is due to Tucker (1964), and the scale types are derived of course from Stevens (1946, 1951).

The main difference between Coombs' and our approach is that we attempt *separate* taxonomies of data and of models, whereas Coombs argued that data cannot be classified independently of the model to which those data are referred, so that the very same data (observations, in Coombs' terms) may fit into different quadrants (or octants) of his schema, depending on which model is assumed. Our attempt to separate the classification of data and of models may be only partially successful, since there is certainly a strong connection between type of data and of model. (There is only a limited class of models suitable for any specific type of data.) We shall nevertheless attempt to maintain the distinction wherever possible.

When one considers the highly important aspect of scale typology, it could be applied exclusively to the data (a la Stevens) or to the model (as suggested by Guttman 1971), but our view is that the scale typology is decidable separately for the data and for the model. For the former, it seems self-evident that some tasks *ask* the subject to adhere to certain scale types (e.g. sorting versus magnitude estimation). However, there can be little doubt that during the years covered by this review, far greater practical emphasis has been placed on incorporating the transformations underlying the scale typology into the model.

The advantages of maintaining the typology for both data and model are apparent from consideration of Shepard's (1972a, Chang & Shepard 1966) approach that embodied an exponential decay fitting procedure in a metric multidimensional scaling analysis. Unless such a transformation (characteristic of many models of forgetting or confusions) can be accommodated by the scale typology, then it must be claimed that Shepard's analysis produced a new and distinct type of scale. We find it more parsimonious to view the data as ordinal and Shepard's analysis/model as interval with a transformation included. We could explicitly include scale type as a property of such models; however, for the present we are including scale type only as a property of the data. The current version of our new schema is presented below:

I. Properties of Measurement Data

Definitions

A mode is defined as a particular class of *entities.* Modes will be denoted by capital letters A, B, C, . . . etc. Entities could be, for example, subjects, stimuli, test items, occasions, experimental conditions, geographical areas, or components of a "multiattribute stimulus." Particular members of the class of entities corresponding to a mode are denoted by subscripts; e.g. A_i, $i = 1,2,\ldots,S$ could denote S subjects.

An N-way array is defined as the cartesian product of a number of modes, some of which may be repeated. For example, an array associated with three-way multidimensional scaling might be of the form AxBxB, where A denotes subjects, and B stimuli. An element of the array is a particular value of this cartesian product [i.e. a combination of particular members of the modes; e.g. (A_i,B_j,B_k)]. A *data array* is an assignment of scale values to some or all elements of the array with possible replications.

Having established these definitions, the taxonomy of data arrays follows straightforwardly, as outlined below:

A. Number of modes
   1. One mode
   2. Two modes
   3. Three or more modes

B. Power of a given mode
   A mode's power is the number of times the mode is repeated in the N-way table.
   1. Monadic data (e.g. single stimulus data, as from an absolute judgment task). Power = 1
   2. Dyadic data (e.g. proximities data). Power = 2
      a. Symmetric
      b. Nonsymmetric
   3. Polyadic data (e.g. judgments of homogeneity of sets of three or more stimuli, or similarity of or preference for "portfolios" of a number of items from the same set). Power $\geqslant 3$
   (Note: In principle each mode could be of power greater than one. In practice only the "stimulus" mode commonly has power greater than one.)

C. Number of ways, defined as total number of factors, whether repeated or not, defining the data array; N if table of data is N-way (exclusive of replications, which are not usually thought of as defining a separate mode or way unless there is a structure on the replications *and* the replications "mode" is explicitly included in the model). (Note: The number of ways is clearly redundant with the first two data properties, since it is just the sum over modes of the power of each mode. However, we find it convenient to include this redundant property explicitly in our schema.)

D. Scale type of data (after Stevens, but with some additions)
   1. Nominal
   2. Ordinal
   3. Interval
   4. Ratio (sometimes called "interval with rational origin")
   5. Positive ratio
   6. Absolute

We have added to Stevens' four scale types what is sometimes called the "interval with rational origin" (which we simply call "ratio") that can be viewed as a ratio scale admitting negative as well as positive values (and, of course, zero), and the "absolute" scale (e.g. Zwislocki 1978), in which no transformation whatsoever is allowed. At the suggestion of Amos Tversky, we are relabeling Stevens' "ratio scale" the "positive ratio scale" (i.e. a ratio scale that allows only non-negative values).

E. Conditionality of data
  1. Unconditional data
  2. Row or column conditional data (Coombs 1964)
  3. Matrix conditional data
  4. Other types of conditional data

F. Completeness of data
  1. Complete data
  2. Incomplete data

G. Number and nature of replications
  1. Only one data *set* comprising the data array
  2. Two or more data sets
    a. Same scale type for each replication
    b. Different scale types for different replications

II. Properties of Multidimensional Measurement Models

A. Type of geometric model
  1. Spatial
    a. Distance models
      i) Euclidean
      ii) Minkowski-p (or l_p) metrics
      iii) Riemannian metrics
      iv) Other non-Euclidean metrics
    b. Scalar product (or projection) models
  2. Nonspatial (discrete set-theoretic or graph-theoretic models)
    a. Nonoverlapping classes (partitions) (e.g. standard clustering methods)
    b. Overlapping classes (e.g. Shepard-Arabie ADCLUS model)
    c. Hierarchical tree structure
    d. Multiple tree structure
  3. Hybrid models (mixtures of continuous and discrete structure)
    a. Mixture of (single or multiple) tree structure and spatial structure
    b. Mixture of class structure (overlapping or nonoverlapping) and spatial structure
      i) Dimensions that generalize over all classes
      ii) Some class-specific dimensions
      iii) Both of the above

B. Number of sets of points in space (or other structure)
  1. One set
  2. Two sets
  3. More than two sets

C. Number of spaces or structures (and their interrelations)
  1. One space or structure (e.g. two-way MDS)
  2. Two spaces or structures [e.g. stimulus (or other "object") space and subject (or other "data source") space in three-way or individual differences MDS]
  3. More than two spaces or structures

D. Degree of external constraint on model parameters
  1. Purely "internal" solutions in which all model parameters are unconstrained
  2. Various kinds of linear, ordinal, or other constraints on specific parameters of model
  3. "External" models, in which one or more spaces (or other structures), or one or more sets of points in the same structure, is totally fixed

The present survey of multidimensional scaling and related techniques is organized around our taxonomy, but does not conform exactly, owing to space limitations and, of course, uneven progress in various subareas during recent years. We take aspects I-A and I-C ("modes" and "ways" from the properties of data) as the dominant organizing principles. Since we are defining as our goal the imposition of structure on the field as a whole, we shall take somewhat greater liberty than may be usual for *Annual Review* chapters, to cite work that may lie outside the time period we are primarily covering, to cite or refer to unpublished work, or even in some cases work still in progress. We also note that we may underemphasize applications of MDS relative to theoretical and methodological developments. (Some methodological areas are also underemphasized or omitted altogether.) We hope any imbalance that results will be partly corrected by a bibliography now in preparation at Bell Laboratories (also see Bick et al 1977, Nishisato 1978a).

ONE-MODE TWO-WAY DATA

We begin our discussion of MDS data and models with the class of data most frequently encountered: one-mode two-way data, which could otherwise be characterized as two-way dyadic data. These data are typically some form of similarities, dissimilarities, or other proximities data (e.g. measures of association between pairs of stimuli or other objects, frequencies of confusions, second order measures of similarity or dissimilarity derived from standard multivariate or other data, etc). A general overview discussing and interrelating most of the spatial (both distance and scalar product) models and corresponding methods for analysis of such data (as well as two-mode three-way data) is provided by Carroll & Kruskal (1977; see also Carroll & Wish 1974b). Another type of ostensibly dyadic data are so-called "paired comparisons" data depicting preferences or other forms of dominance relations on members of pairs of stimuli. However, such data are seldom utilized in multidimensional (as opposed to unidimensional) scaling. We do not cover paired comparisons data in this section because we view such data *not* as dyadic, but as replicated monadic data (having n-2 missing data values within each replication).

Spatial Distance Models (for One-mode Two-way Data)

The most widely used MDS procedures are based on spatial *distance models.* These are geometric models in which the similarities, dissimilarities, or other proximities data are assumed to relate in a simple and well-defined manner to recovered *distances* in an underlying *spatial* representation. If the data are interval scale, the function relating the data to distances would generally be assumed to be inhomogeneously linear; that is, linear with an additive constant as well as a slope coefficient. Data that are interval scale or stronger (ratio, positive ratio, or absolute) are called *metric* data, while the corresponding models and analyses are collectively called *metric* MDS. In the case of ordinal data, the functional relationship is generally assumed to be monotonic—either monotonic nonincreasing (in the case of similarities) or monotonic nondecreasing (for dissimilarities). Ordinal data are often called nonmetric data, and the corresponding MDS models and analyses are also referred to as *nonmetric* MDS. The distinction between metric and nonmetric is based on the presence or absence of metric properties in the data [*not* in the solution, which almost always has metric properties; Holman (1978) is an exception].

Following Kruskal's (1964b, 1965) innovative work in monotone regression (as the basic engine for fitting most of the ordinal models considered in this review), first devised by Ayer et al (1955), there has been much activity in this area of statistics. In addition to Shepard's (1962a,b) early approach and Guttman's (1968) rank image principle, there have also been alternative and related methods proposed by Barlow et al (1972), Johnson (1975), Ramsay (1977a), Srinivasan (1975), and de Leeuw (1977b). A provocative comparison between the approaches of Kruskal (1964b) and Guttman (1968) is given by McDonald (1976), and the two methods are subsumed as special cases of Young's (1975b) general formulation. Shepard & Crawford (1975, Shepard 1974) and Goldstein & Kruskal (1976) have developed techniques for imposing various constraints on ordinal regression functions.

The range of types of data to which MDS analyses are applicable has recently been extended through the use of *nominal* scale techniques of regression (Nishisato 1971, Hayashi 1974, Young et al 1976, Bouroche et al 1977, Young & Null 1978), as found in the ALSCAL program (discussed below) of Takane et al (1977).

UNCONSTRAINED SYMMETRIC DISTANCE MODELS (FOR ONE-MODE TWO-WAY DATA) Although one of the most intensely developed areas in recent years has been the treatment of nonsymmetric data (discussed in detail below), it is still true that most of the extant data relevant to MDS are symmetric, owing in part to the previous lack of models allowing for nonsymmetric data. Therefore, we first consider recent developments in the scaling of symmetric data, i.e. where the proximity of *a* to *b* is assumed identical to that obtained when the stimuli are considered in the reverse order.

Euclidean and Minkowski-p metric The most widely assumed metric in MDS work is the Euclidean, in which the distance between two points i and j is defined as

$$d_{ij} = [\sum_{r=1}^{R}(x_{ir} - x_{jr})^2]^{1/2}$$

(where x_{ir} and x_{jr} are the rth coordinates of points i and j, respectively, in an R-dimensional spatial representation). Virtually all two-way MDS procedures use either the Euclidean metric or the Minkowski-p (or l_p) metric which defines distances as:

$$d_{ij} = [\sum_{r=1}^{R}|x_{ir} - x_{jr}|^p]^{1/p} \qquad (p \geqslant 1)$$

and so includes Euclidean distance as a special case in which $p = 2$.

The program KYST (Kruskal et al 1973, 1977) was christened with an acronym based on the names of Kruskal, Young, Shepard, and Torgerson. KYST is a combination of what were regarded by many as the preferred features of Kruskal's (1964a,b) MDSCAL and Young & Torgerson's (1967) TORSCA, and also includes the new feature of "constrained" or "external" analyses in which a subset of the stimuli are given fixed coordinates by the user while the remaining stimuli are mapped into the constrained configuration.

Other algorithms include the Guttman-Lingoes family of two-way "Smallest Space Analysis" MDS procedures (Lingoes 1972, 1973; see also Lingoes 1977) and Roskam's (1975) related series of programs. An informative discussion comparing several of these different algorithmic approaches to MDS is given by Kruskal (1977a; see also de Leeuw & Heiser 1980). Other techniques have also been devised by Young (1972), Johnson (1973) and Hubert & Busk (1976).

Two of the most valuable algorithmic developments in unconstrained two-way (*and* three-way) MDS within the period covered by this review are the Takane et al (1977) ALSCAL procedure and Ramsay's (1977b) MULTISCALE. ALSCAL (for Alternating Least squares SCALing) differs from previous two-way MDS algorithms in such ways as (*a*) its loss function, (*b*) the numerical technique of alternating least squares (ALS) used earlier by Carroll & Chang (1970) and devised by Wold (1966; also see de Leeuw 1977a, and de Leeuw & Heiser 1977), and (*c*) allowing for nominal scale (or categorical), as well as interval and ordinal data. Both ALSCAL and MULTISCALE are also applicable to two-mode three-way data, and both programs will be considered again under spatial distance models for such data.

MULTISCALE (MULTIdimensional SCAL[E]ing), Ramsay's (1977b; also see Ramsay 1975) maximum likelihood based procedure, although strictly a metric (or linear) approach, has statistical properties which make it potentially much more powerful as both an exploratory and (particularly) a confirmatory data analytic tool. MULTISCALE, as required by the maximum likelihood approach, makes very explicit assumptions regarding distribution of errors, and about the relationship of parameters of this distribution to parameters defining the underlying spatial representation. One such assumption is that the dissimilarity values δ_{ij} are log normally distributed over replications, but other distributional assumptions are also allowed.

The major dividend from Ramsay's (1978) strong assumptions is that the approach enables statistical tests of significance that include, for example, assessment of the correct dimensionality appropriate to the data, via an asymptotically valid chi square test of significance. Another advantage is the resulting confidence regions for gauging the relative precision of stimulus coordinates in the spatial representation. The chief disadvantage is the very strong assumptions that must be made for the asymptotic chi squares and/or confidence regions to be valid. Not least of these is the assumption of ratio scale dissimilarity judgments. In addition, there is the assumption of a specific distribution (log normal or normal with specified parameters) and of statistical independence of the dissimilarity judgments.

Applications and theoretical investigations of the Euclidean, Minkowski-p, and other intradimensionally subtractive and interdimensionally additive metrics (for one-mode two-way symmetric data) While the Euclidean distance formula has certain computational conveniences to recommend it as a statistical model, only within recent years has the formula been viewed as a possible contender for a psychological model. Relevant research has followed along three lines, the earliest of which stems from Beals et al (1968), who provided a set of testable axioms underlying a wide class of distance metrics (including Euclidean and Minkowski-*p*) as a psychological model. Two of these conditions, intradimensional subtractivity and interdimensional additivity, were extensively violated in the perception of similarity of rectangles in Krantz & Tversky (1975). A very thorough follow-up by Wiener-Ehrlich (1978) also found an interaction between dimensions for rectangle stimuli. However, for stimuli that were Munsell papers varying along the "separable" dimensions of area and brightness, she found that her data did satisfy the relevant axiomatic conditions. Related research has also been reported by Monahan & Lockhead (1977), Schönemann (1977), Zinnes & Wolff (1977), and Chipman & Noma (1978). At present it seems that no general conclusion can be drawn from this approach to the validity of distance models, but there is certainly no strong support forthcoming.

A major boost for the plausibility of distance models was provided by the elegant work of Rumelhart & Abrahamson (1973), who presented data consistent with a model in which the traditional analogy $a : b :: c : x$ implies a parallelogram in a metric space. The study also established that subjects' solutions to certain types of analogical problems were in fact successfully predicted by an independently obtained MDS solution. Other experiments in which the parallelogram rule was verified were designed by Sternberg (1977). Also, a scaling algorithm which takes as input judgments assumed to fit the model (and thus implying linear vector equations a la Rumelhart and Abrahamson) was devised by Carroll & Chang (1972b).

The third stage for questioning the viability of distance models for psychological similarity was set by important papers by Tversky (1977) and Tversky & Gati (1978). While space limitations prohibit an adequate summary or discussion of those papers, the main challenges to distance models were (*a*) questioning of the minimality ($d_{ii}=0$) and (*b*) symmetry ($d_{ij}=d_{ji}$) conditions of the metric axioms, and (*c*) arguments advocating discrete features as opposed to continuous dimensions as the underlying basis of psychological similarity. Several of these challenges have been eloquently answered by Krumhansl (1978) and will be considered below.

Somewhat oblivious to the validity of the preceding studies, nonmetric (two-way) scaling has continued to grow in popularity, and we are able to mention only a small proportion of the applications in recent years. Scaling has provided representations of structure in memory (Wexler & Romney 1972, Arabie et al 1975, Shepard et al 1975, Holyoak & Walker 1976, Ebbesen & Allen 1979). Studies by Shoben (1976, p. 372) and Friendly (1977, p. 206) have demonstrated the utility of the often overlooked option in KYST (Kruskal et al 1973, 1977) of differentially weighting the stimuli being scaled. The relevance of scaling to memory and other experimental aspects of educational research has been reviewed by Subkoviak (1975).

Many applications of scaling to perceptual phenomena have been covered by Fillenbaum & Rapoport (1971), Carroll & Wish (1974b), Indow (1974), and Gregson (1975). Other scaling studies of visual processes include Heider & Olivier (1972) and Reed (1972). The substantive importance of determining the correct dimensionality of a scaling solution was underscored by the comments of Rodieck (1977) on the investigations of Tansley & Boynton (1976, 1977). Multidimensional scaling has also been found increasingly useful in olfaction (Moskowitz & Gerbers 1974, Schiffman & Dackis 1976). In psychoacoustics, two-way scaling has continued to play a prominent role, with examples provided by Shepard (1972a), Wang et al (1978), Cermak (1979), and Krumhansl (1979).

Seriation is a term which comes from archaeology and refers to unidimensional representation of a set of objects, where the dimension in question is usually time, so that the result is a chronological ordering of those entities. In several ways, seriation defies our taxonomy, although the original data are typically two-mode two-way and nonsymmetric. An example in archaeology would be an incidence matrix of artifacts by sites, with the objective of separately ordering (i.e. seriating) the objects corresponding to each mode. A corresponding problem in psychology considers a subjects-by-item response matrix (Hubert 1974a).

In spite of the description just given of the basic data structure, the actual analysis typically begins with one-mode two-way symmetric data that are analyzed by KYST (Kruskal et al 1973, 1977) or some variant of that program. An adequate summary of developments culminating in this practice would greatly exceed the length of this chapter; for an overview, see Hubert (1974b, 1976) or Arabie et al (1978). The central idea of using KYST or related programs to get a Euclidean *two*-dimensional representation from which the (one-dimensional) seriation is inferred is due to Kendall (1970, 1975; see Shepard 1974, pp. 385–89 for an example). Refinements in this technique can be found in chapters of Hodson et al (1971), especially the papers by Kendall (1971a,b), Sibson (1971), and Wilkinson (1971). Important work has also been done by Kupershtokh & Mirkin (1971), Wilkinson (1974), Graham et al (1976), Hubert & Schultz (1976a), Baker & Hubert (1977), and Defays (1978). The applicability of seriation to substantive problems in psychology is cogently illustrated by Coombs & Smith (1973) and Hubert & Baker (1978).

Continuing theoretical interest in non-Euclidean Minkowski-p ($p \neq 2$) metrics is evinced in papers by Fischer & Micko (1972), Carroll & Wish (1974b), Shepard (1974), Arabie et al (1975), and Lew (1978). While it is not uncommon to find articles oblivious to the difficulties of local minima in non-Euclidean nonmetric scaling, the problems have been documented by various authors and appear not to be limited to specific scaling programs. Arabie & Boorman (1973) reported extensive local minima for non-Euclidean metrics using Kruskal's MDSCAL, and Ramsay (1977b, p. 255) found similar difficulties with his MULTISCALE.

Perhaps the first effort explicitly to overcome some of these drawbacks was by Arnold (1971), who obtained Euclidean solutions which were then used as rational initial configurations for Minkowski-p ($p \neq 2$) metrics. The latter solutions served iteratively as initial configurations for p-values increasingly discrepant from 2, in search of the p-value for which stress was least for a given dimensionality. In unpublished work, some of which is described by Shepard (1974), Arabie replicated Arnold's results, and found that Arnold's approach generally worked well for various data sets, if the declared dimensionality exceeded 2. For reasons still not understood, the Arnold strategy appears not to work well in two-dimensional spaces, where Arabie instead used many different random initial configurations. Also, Shepard (1974) has cautioned that Kruskal's (1964a,b) measure of badness-of-fit, stress, may not be comparable across different Minkowski p-values when the data are heavily tie-bound.

The extent to which Shepard's caveat is applicable to real data is presently unknown. However, it is clear that obtaining a lower stress value for a non-Euclidean metric is a necessary but not sufficient condition for declaring data to be non-Euclidean. Shepard & Arabie (1979, p. 115) presented a city-block solution possessing a least stress value for that particular Minkowski metric as well as a substantive interpretation for the unrotated axes. Another instance of a best-fitting city-block metric was given by Wiener-Ehrlich (1978, p. 405).

Metrics other than Euclidean or Minkowski-p There have been some interesting developments in MDS involving more general metrics. Perhaps the most general of these is Holman's (1978) "completely nonmetric" MDS procedure. This approach can in some ways be viewed as an explicit algorithm to accomplish what Coombs (1964) attempted more heuristically and less algorithmically in his "nonmetric scaling" approach. That is, Holman's approach is nonmetric both vis-à-vis the data *and* the solution (the latter in the sense that only the rank order of coordinate values are defined).

Recently considerable interest has focused on another class of metrics—the Riemannian metrics. Motivated largely by Luneburg's (1947, 1950) theory, Indow (1974, 1975, 1979) has made various attempts to fit Riemannian metrics with constant negative curvature to data relevant to the geometry of visual space (e.g. judgments of distances among fixed light sources), but has not developed an MDS algorithm involving a Riemannian metric. The first attempt at Riemannian multidimensional scaling was an approach by Pieszko (1975), who first used "classical" metric MDS (Torgerson 1958) to fit a configuration, limited to two dimensions, to the data and then obtained a very rough approximation for Riemannian distances defined on that configuration. Lindman & Caelli (1978) have criticized the inappropriateness of Pieszko's global approximation, which is only valid locally. Those authors were the first to produce a genuinely Riemannian (metric) MDS procedure, for Riemannian metrics of constant curvature. In some unpublished work, Caelli, Carroll, and Chang have extended this approach to include Riemannian metrics of positive nonconstant curvature.

More general Riemannian metrics can also be considered, involving geodesic metrics defined in very general nonlinear surfaces (or manifolds) embedded in high-dimensional Euclidean space. An interesting paper by

Shepard (1978) describes a number of perceptual (and/or judgmental) phenomena that could be represented in terms of such very general geometric models. Weisberg (1974) provides an urbane discussion of the relevance of a priori structures (and the underlying models) to psychology and related behavioral sciences.

CONSTRAINED SYMMETRIC EUCLIDEAN MODELS (FOR ONE-MODE TWO-WAY DATA) A number of approaches have emerged quite recently that allow the imposition of various kinds of constraints on two-way MDS (distance model) solutions. To date, all such research has involved the case of symmetric data and has been restricted to the Euclidean metric. More recent approaches include: Bentler & Weeks (1978), in which linear constraints (equality of specified pairs of coordinate values or proportionality to given external values) are imposed; Noma & Johnson (1979), in which inequality constraints are imposed on coordinate values (i.e. a given dimension in the solution is constrained to be monotonically related to an external variable); and Borg & Lingoes (1979), in which inequality constraints are imposed on certain distances. Recently de Leeuw & Heiser (1979) have discussed a very general class of algorithms for fitting constrained models of many different kinds. Finally, an approach called CANDELINC (Carroll et al 1976, Green et al 1976, Carroll, Pruzansky & Kruskal 1979) includes as a special case a version of "classical" metric two-way MDS in which a very general class of linear constraints are imposed. (See discussion under two-mode three-way constrained models.)

UNCONSTRAINED NONSYMMETRIC EUCLIDEAN MODELS (FOR ONE-MODE TWO-WAY DATA) A number of approaches exist for analysis of nonsymmetric dyadic data in terms of a Euclidean model. In the analysis of nonsymmetric data, an important general principle is the following: any *nonsymmetric* m-mode n-way data set can be accommodated by a *symmetric* model designed for (m+1)-mode n-way data. The extra mode arises from considering the "rows" and "columns" as corresponding to *distinct* entities, so that each entity will be depicted twice in the representation from the symmetric model. This principle is valid throughout our discussions of nonsymmetric data, and we will therefore not repeat it in subsequent sections.

An alternative, second general principle in the analysis of nonsymmetric proximities data assumes they are row or column conditional (possibly a correct assumption), but employs a model allowing only one set of entities. Thus the model is symmetric, but nonsymmetry is assumed to result from conditionality of the data. Such analyses are possible in MDSCAL-5 and KYST, as well as in a procedure proposed by Roskam (1975) called

MNCPAEX. (See external distance models for two-mode two-way data.) Takane (1979) has produced a nonmetric maximum likelihood approach that allows conditional rank order data. Takane's algorithm is especially interesting because it is simultaneously *parametric* (in the sense that a specific error distribution is assumed) and *nonmetric* (in that the data are strictly ordinal).

Gower (1978) has recently applied unfolding techniques (discussed under spatial distance models for two-mode two-way data) to nonsymmetric dyadic proximities data. In addition, a general approach for decomposing nonsymmetric data matrices has been developed independently by Tobler (1976) and Gower (1977, Constantine & Gower 1978), while another has been proposed by Holman (1979), in which nonsymmetric proximities are analyzed (via nonmetric models) into symmetric proximities and row and/or column bias parameters.

Young's (1975a) ASYMSCAL (for ASYMmetric multidimensional SCALing) provides another approach for analysis of nonsymmetric data. ASYMSCAL allows differential weights for dimensions for either the row stimuli or for the column stimuli, or both. In this respect ASYMSCAL closely resembles a weighted generalization of the unfolding model that will be discussed in the section on unfolding.

Theoretical developments for and applications of nonsymmetric analyses (for one-mode two-way data) Until very recently, asymmetries in a proximities matrix have often been regarded as a nuisance—something to be averaged out or eliminated by various strategies. The recent proliferation of models for asymmetric data has coincided with increased awareness of the psychological importance of asymmetries in proximities data. Tversky (1977) and Tversky & Gati (1978) cite many examples of psychological processes giving rise to nonsymmetric data (see Sjöberg 1972) and leave the reader with the impression that the psychological universe may indeed be more nonsymmetric than symmetric.

Tversky (1977) and Tversky & Gati (1978) develop the argument still further in advocating the superiority of feature-theoretic models to continuous spatial dimensions for representing structure in data (e.g. Gati 1979). However, Krumhansl (1978), drawing extensively on findings from unidimensional psychophysics, has developed a highly ingenious "distance-density" model that assumes similarity is a function of both interpoint distance and the spatial density of other stimulus points in the surrounding region of the metric space. Krumhansl finds support in the literature for various predictions made by her model (also see Podgorny & Garner 1979) and suggests that spatial distance models may still be more relevant to nonsymmetric data than Tversky (1977, Tversky & Gati 1978) argued.

While careful consideration of experimental procedures in order to avoid artifactual asymmetries is still warranted (Janson 1977), current practice clearly pays much greater attention to (and respect for) asymmetries in data; e.g. Cermak & Cornillon (1976), Zinnes & Wolff (1977), Jones et al (1978), Krumhansl (1979). Also, a useful inferential test for symmetry in a proximities matrix has been developed by Hubert & Baker (1979).

Scalar Product (Spatial but Nondistance) Models (for One-mode Two-way Data)

The scalar product between points i and j (b_{ij}) is defined in terms of their coordinates (x_{ir} and x_{jr}, $r = 1,2, \ldots R$) as:

$$b_{ij} = \sum_{r=1}^{R} x_{ir} x_{jr} \quad .$$

Scalar product models are sometimes called "projection models" because the scalar products of a set of points with a fixed point are proportional to the projections of those points onto a vector from the origin of the coordinate system to the fixed point.

After the factor analytic model (not considered in this chapter), probably the most widely known scalar product model for symmetric proximities data is Ekman's "content model." Important articles discussing this class of models, whose popularity has declined within the period covered by this review, are Eisler & Roskam (1977) and Sjöberg (1975). The latter argues strongly against the content model, in favor of the more widely accepted class of distance models for proximities data.

Other scalar product symmetric approaches include Guttman and Lingoes' SSA-III (Lingoes 1972, 1973) and certain options in Young's (1972) POLYCON (for POLYnomial CONjoint analysis). Both programs are *nonmetric* factor analytic procedures applicable to symmetric data, usually but not necessarily correlations or covariances. Further discussion of these models will be found under unconstrained scalar product models for two-mode two-way data.

In considering scalar product models for *nonsymmetric* dyadic data, there is Harshman's (1975, 1978) metric procedure DEDICOM (DEcomposition into DIrectional COMponents), which can also handle two-mode three-way data (see below). The "strong" case of the model assumes a common set of dimensions for the rows and columns, so that the model is in that sense symmetric. Asymmetries are accounted for in this model by a set of indices of "directional relationship" which indicate the degree to which each dimension affects each other dimension. One way of viewing the strong DEDICOM model is as a special case of the factor or components analysis model in which factor loadings and factor scores are constrained to be linearly related to each other. (The "weak" model is precisely equivalent to the factor or components analysis model.)

A model involving a geometrically interesting generalization of scalar products (defined only for two or three dimensions, however) has been formulated by Chino (1978) for one-mode two-way nonsymmetric data.

Nonspatial Distance Models (for One-mode Two-way Data)

A development that has occurred almost entirely within the period covered by this review is that of nonspatial or discrete models for proximities data. Of course the vast area of clustering has long allowed such representation of proximities, but the solutions have infrequently been viewed as realistic psychological models for proximities data. Moreover, as is true with factor analysis, the clustering literature is much too vast to be covered here, so we refer the reader to Sneath & Sokal (1973), Hubert (1974c), Hartigan (1975), and Blashfield & Allenderfer (1978) for relevant reviews.

Backtracking somewhat, we first consider an approach by Cunningham & Shepard (1974) that is, in fact, *neither* spatial nor nonspatial. This "nondimensional" scaling approach transforms the data so as to satisfy the metric condition of the triangle inequality. The method is useful primarily in converting ordinal proximities into ratio scale distance estimates, which could then be used as data for various *metric* analyses, or for determining the form of the function relating proximities to distances.

One nonspatial model that assumes a discrete geometric model is the Shepard & Arabie (1979, Arabie & Shepard 1973) ADCLUS (for ADditive CLUStering) model. ADCLUS assumes that similarities data can be represented in terms of discrete but possibly overlapping classes or clusters, and that each of these clusters has a non-negative weight (although an additive constant interpretable as the weight for the cluster corresponding to the complete set is not so constrained). The predicted similarity for any pair of stimuli is just the sum of the weights across the clusters containing that pair of stimuli. Formally stated, s_{ij} is approximated by

$$\hat{s}_{ij} = \sum_{r=1}^{R} w_r p_{ir} p_{jr}$$

where s_{ij} is similarity of stimuli (or other objects) i and j, w_r is the weight for the rth class, R is the number of classes, analogous to the number of dimensions in various spatial models, and p_{ir} is a binary (0,1) class membership function ($p_{ir} = 1$ *iff* stimulus i is a member of class r, and 0 otherwise). This model is formally equivalent to the factor analytic model (without communalities) for correlations or covariances, except for the constraint that the p_{ir} be restricted to the discrete values of 0 or 1. In addition, ADCLUS represents a special (symmetric) case of Tversky's (1977) general features model of similarity, and is in fact the only case for which an analytic procedure is currently operational.

Arabie & Carroll (1978) have provided a different *algorithm* called MAPCLUS (for MAthematical Programming CLUStering) for fitting the ADCLUS model, since the algorithm used in the Shepard-Arabie (1979) program was very expensive computationally and otherwise unwieldy. Moreover, the MAPCLUS approach is easily generalized to fit the three-way model, called INDCLUS (Carroll & Arabie 1979).

Tree structures comprise another interesting class of discrete geometric models. For a given tree structure there are at least two (and in some cases three) types of metrics that can be defined on the stimuli. In this representation, the stimuli are represented as nodes of the tree, either terminal nodes only or both terminal and nonterminal. One of the two classes of metrics is the *ultrametric* (Hartigan 1967, Jardine et al 1967, Johnson 1967), in which "heights" are associated with nonterminal nodes of the tree, and "distance" between any two nodes is defined as the "height" of the first nonterminal node at which the two are linked.

An interesting relationship between ultrametric and Euclidean metrics (see above) was formally derived by Holman (1972), who showed that a Euclidean representation of "ultrametric data" requires n-1 dimensions, where n is the cardinality of the largest subset of stimuli satisfying the ultrametric inequality. While this demonstration has somewhat limited applicability to data containing error, Holman's (1972) result should help dispel a lingering misconception from the factor analytic tradition, namely that distance-based scaling models are legitimately serviceable as a clustering method; they are not (cf Kruskal 1977b).

A second metric, after the ultrametric, has been given a variety of names, and the same is true for the resulting representations. The metric is simply defined as the shortest path in terms of lengths of the "branches" or "links" connecting adjacent nodes in the tree. For a tree structure there is only *one* path connecting any pair of nodes, so the shortest path is trivially defined as the length of that unique path. This metric was designated as "path length" and the associated trees as "path length trees" by Carroll & Chang (1973), Carroll (1976), and Carroll & Pruzansky (1975, 1980). Alternative algorithms for fitting the metric, as well as relevant applications, are given by Cunningham (1974, 1978) and Sattath & Tversky (1977). Other important references include Bunemann (1971, 1974) and Dobson (1974). We note that some of these authors have also given other names to this metric and/or trees on which it is defined.

Carroll & Chang (1973) also allowed a third type of metric, namely a synthesis of ultrametric and path length metric, in which distances were defined as the sum of the path length and a height value associated with the "least common ancestor" node. It can be shown that this "combined" metric can be meaningfully defined only in the case (allowed by Carroll & Chang 1973) of trees in which the stimuli or other objects are associated with at least some nonterminal as well as terminal nodes.

The approach of Carroll & Pruzansky (1975, 1980; see also Carroll 1976) utilized mathematical programming techniques, analogous in some ways to those used in the Arabie-Carroll (1978) MAPCLUS approach, to fit either ultrametric or path length trees to proximities data via a least squares criterion. The essential new feature of the Carroll-Pruzansky approach, however, is the generalization to *multiple* tree structures, for which proximities data are represented by composite distances summed over distances (either ultrametric or path length) from two or more trees. Carroll and Pruzansky applied this approach to a number of data sets, with interpretable results, and conjectured that there may be a relatively well-defined sense in which a single tree structure is approximately equivalent to a two-dimensional spatial structure (cf Sattath & Tversky 1977).

A *constrained* nonmetric analysis in terms of (single) path length tree structure models has been described by Roskam (1973), which allows such options as constraining certain branch lengths to be equal. Constrained analyses are also possible by using appropriate options in most of the procedures designed for unconstrained fitting of the ADCLUS or tree structure models.

Hybrid Distance Models (for One-mode Two-way Data)

By hybrid geometric models we denote models that in some way combine continuous spatial structure of the type classically associated with MDS with discrete nonspatial structure such as assumed in ADCLUS, tree structure, other more general graph-theoretic structures, or combinations of these. Carroll & Pruzansky (1975) have produced the only approach known to us of "wholistic" fitting of a hybrid model, where both components are *simultaneously* fitted to the data. This hybrid model combines a tree structure component (either single or multiple) with an R-dimensional spatial component, and uses an alternating least squares procedure for fitting the model. Very good results were obtained in such a hybrid analysis of some kinship data obtained from a sorting task by Rosenberg & Kim (1975). We expect to see other hybrid models formulated and the associated analytic procedures implemented in the future (Carroll 1975, 1976).

TWO-MODE TWO-WAY DATA

We now consider two-way data in which the two ways correspond to distinct modes (e.g. subjects and stimuli). The data array in this case will correspond (in general) to a rectangular matrix which is generally nonsymmetric (even in the case when, by coincidence or design, the matrix is square).

Spatial Distance Models (for Two-mode Two-way Data)

The principal distance model for studying individual differences in preference judgments (as a case of two-mode two-way data) is the unfolding model. This approach was originally formulated by Coombs (1950), with a multidimensional generalization provided by Bennett & Hays (1960). The hallmark of this model is that both stimuli and subjects' ideal points are simultaneously mapped into the same spatial configuration. As such, this approach constitutes what we have called an *internal* (unconstrained) analysis of preference data. The original developments by Coombs (1950) and Bennett & Hays (1960) assumed the data were ordinal and conditional, the latter by subjects for subjects by stimuli preference data. Subsequent proposals and corresponding computer programs have allowed for interval (metric) and/or unconditional data. Also, more recent procedures include *external* analyses, which are constrained in that the stimulus space is given a priori, while the subjects' ideal points are fitted on the basis of the preference data.

It should be emphasized that the unfolding model is not limited only to subjects by stimuli preference data, but may be applied to any two-mode two-way data matrix for which a distance model is appropriate. Recall, in particular, the first principle given above for representing nonsymmetric data. For a general and more detailed discussion of unfolding models, see Carroll (1972, 1980).

INTERNAL (UNCONSTRAINED) DISTANCE MODELS (FOR TWO-MODE TWO-WAY DATA) Procedures that allow internal unfolding analysis include KYST (Kruskal et al 1973, 1977) as well as its predecessors MDSCAL-5 and TORSCA-9 (cited above), Guttman and Lingoes' SSAR-I and SSAR-II procedures (Lingoes 1972, 1973), and a procedure by Roskam (1971; see also Lingoes & Roskam 1973). Of these, only KYST and MDSCAL-5 (or 6) use an appropriate loss function and/or allow use of a loss function (stress "formula two") with a variance-like normalization (for conditional analyses) or *metric* unfolding options (for unconditional analyses). Those two programs thus are the only theoretically valid implementations, since trivial "degenerate" solutions (with a zero value of the loss function) occur when a loss function like stress formula one (having a normalizing factor resembling the sum of squared distances) is used, or when nonmetric conditional analysis is done (irrespective of what loss function is used). The rationale for a loss function like stress formula two can be found in Kruskal & Carroll (1969; see also Carroll 1980). Programs other than KYST and MDSCAL-5 sometimes yield what appear to be good solutions despite this theoretical problem, but such findings are generally the results of convergence to substantively acceptable *local* minima, rather than the *global* minimum corresponding to a degenerate solution. An example of the latter is a configuration in which the entities corresponding to the two modes are each mapped into a single point.

In addition to the metric (internal or external) analyses discussed above, there is a metric unfolding procedure (Schönemann 1970) that attains an analytic internal solution for a very strong case of the unfolding model. Carroll & Chang's PREFMAP-2 (1971, Chang & Carroll 1972), which is primarily designed for external analyses (as discussed below; see also Carroll 1980), also allows an internal solution for a model similar to but slightly more general than Schönemann's (1970). Schönemann & Wang (1972) combine the metric unfolding model with the Bradley-Terry-Luce choice model (Luce 1959), to produce a stochastic unfolding approach that is applicable to paired comparisons data. The MDPREF model, which can be viewed as a special case of the unfolding model [in which the subjects' ideal points are infinitely distant from the stimulus points (Carroll 1972, 1980)] will be discussed in the section on scalar product models.

EXTERNAL (CONSTRAINED) DISTANCE MODELS (FOR TWO-MODE TWO-WAY DATA) *External* analyses in terms of the unfolding model (and some of its generalizations) are provided by the PREFMAP procedure of Carroll & Chang (1967; see also Chang 1969 and Carroll 1972, 1980), by KYST (Kruskal et al 1973, 1977), as well as other programs described below. PREFMAP (and its successor PREFMAP-2, described in Carroll 1980) is based on a general linear least squares approach involving quadratic regression procedures, and allows both metric and nonmetric options.

PREFMAP and PREFMAP-2 also allow fitting of two models more inclusive than the standard unfolding model. One of these models, for "weighted unfolding," allows a more general weighted Euclidean metric, with a different pattern of dimension weights as well as different location of ideal points for each subject. A second generalization allows the most general form of Euclidean metric, defined by a different quadratic form for each subject, thus allowing a different rigid (or orthogonal) rotation of the reference frame for individual subjects, followed by differential weighting of the resulting idiosyncratically defined dimensions. An alternative strategy for implementing nonmetric external unfolding analyses is given by the linear programming approach of Srinivasan & Shocker (1973), which also includes non-negativity constraints for the dimension weights. The same constraints are provided in a metric procedure using quadratic programming described by Davison (1976).

During the years covered by this chapter, substantive applications of both internal and external unfolding include Levine (1972), Coombs et al (1975), Davison (1977), Seligson (1977). Coombs & Avrunin (1977) provided a theoretical derivation of the unfolding model from fundamental psychological principles. There also have been several studies making extensive comparisons (with interesting psychological results) of the structures found when subjects give similarity as well as preference judgments for the same stimulus domain (Cermak & Cornillon 1976, Nygren & Jones 1977, Sjöberg 1977, 1980; also see Carroll 1972 for an early discussion of this question). A new methodological approach for combining proximities with preference (and possibly other rating scale) data has been discussed by Ramsay (1979b). The corresponding model lies somewhere between our categories of "internal" and "external" models.

Scalar Product Models (for Two-mode Two-way Data)

By far the dominant class of models for two-mode two-way data are scalar product models, which include factor analysis and principal components analysis. One approach seeks to fit the score matrix, another to fit correlations or covariances; Kruskal (1978) refers to these as the direct and indirect approaches, respectively.

INTERNAL (UNCONSTRAINED) MODELS (FOR TWO-MODE TWO-WAY DATA) The program SSA-III (Lingoes 1972, 1973) can be viewed as a form of nonmetric factor analysis differing markedly in rationale from the Kruskal-Shepard (1974) variety of nonmetric factor analysis. SSA-III generally assumes correlations or covariances (but can use other proximities data) and seeks a representation involving vectors such that the scalar products between vectors reproduce the order of the *proximities.* In contrast, the Kruskal-Shepard approach starts with a general rectangular (two-mode as well as two-way) data matrix, and seeks a representation in terms of *two* sets of vectors such that the scalar products reproduce (as well as possible) the conditional rank orders (within one of the two modes) of the *scores.* Thus, Kruskal and Shepard's method uses the direct approach, while Guttman and Lingoes' SSA-III or Young's POLYCON uses the indirect approach. The theoretical rationale of the latter approach is less clear because, aside from Guttman's simplex structure, it is difficult to envision conditions where correlations or covariances are only defined ordinally.

In passing we would like to note that when the Kruskal & Shepard (1974) method is applied to two-mode two-way data, it is often expedient to depict the objects of one mode as vectors and the other as points. This representation has various advantages over the more conventional display of both modes as points in a joint space, particularly when the data are conditional with respect to the mode represented by vectors.

A particular type of data to which a factor or component analytic type of model has been very usefully applied comprises a subjects by stimuli matrix of preference (or other dominance) data. In this case the scalar product or projection model has come to be known as the "vector model" because of the very convenient pictorial representation of stimuli as points and subjects as vectors. We view this technique as the "right" way to depict such solutions, since in the case of such data, the matrices are conditional with respect to *subjects.* Thus, the order of projections of (stimulus) points onto a (subject) vector, but not that of vectors onto a point, is meaningfully defined, both in the data and in the geometric representation.

Tucker (1960) and Slater (1960) were the first to propose (independently) somewhat limited versions of such a model for preferences (see also Nishisato 1978b). Probably the most widely used method of analysis involving this model is Carroll & Chang's (1964; see also Carroll 1972, 1980) MDPREF (for MultiDimensional PREFerence analysis), which is actually a special type of factor analysis of either a derived or given preference score matrix. While MDPREF applied to paired comparisons preference data is computationally a metric technique, there is a reasonable index of *ordinal* agreement with the paired comparisons preference data which is optimized by this procedure (Carroll 1972).

It is possible, at least in principle, to effect a multidimensional analysis of "classical" paired comparisons data, in which the paired comparisons judgments are aggregated over different subjects or over replications for a single subject. As argued earlier, the result of such preprocessing can be viewed as replicated two-mode two-way data. A multidimensional model for such a matrix, called the "wandering vector" model, is discussed by Carroll (1980). de Leeuw & Heiser (1979) independently proposed a mathematically equivalent model based on Thurstone's Case I model.

EXTERNAL (CONSTRAINED) SCALAR PRODUCT MODELS (FOR TWO-MODE TWO-WAY DATA) In scalar product external models for two-mode two-way data (as with external unfolding models), one set of points is fixed and the other "mapped in." In the case of conditional data, it is almost always the points corresponding to the conditional mode (the one typically represented as vectors) that are mapped in. One metric means of implementation is multiple linear regression, where the regression coefficients (possibly after some normalization) define the coordinates of the second set of points or vectors. In the case of nonmetric data, some form of what has variously been called nonmetric, ordinal, or monotonic multiple linear regression is necessary. Carroll & Chang's (1971, Chang & Carroll

1972) PREFMAP and PREFMAP-2 both provide metric and nonmetric options for such mapping.

One class of models and methods not usually viewed in this way, but which *can* be characterized as external analysis in terms of a scalar product or vector model (see discussion below and Carroll 1980), is the class including approaches variously called conjoint measurement (Luce & Tukey 1964), functional measurement (Anderson 1974, 1977), and/or conjoint analysis (Green & Wind 1973, Green & Srinivasan 1978). While all three approaches allow more general models, in the most widely known and used versions of these three closely related approaches, a simple additive model is assumed to relate a (metric or nonmetric) dependent variable to a set of qualitative independent variables that form a (complete or fractional) factorial design. In conjoint measurement, the dependent variable is always assumed to be ordinal, in functional measurement it is usually but not always assumed metric, while conjoint analysis includes both metric and nonmetric alternatives.

The additivity analysis central to these three approaches can be viewed as fitting a "main effects only" analysis of variance model to the data either metrically (via classical ANOVA procedures) or nonmetrically. Such additivity analysis can be viewed as an external one in terms of a scalar product model by expressing the main effects ANOVA model in the now widely known form of a multiple linear regression model with appropriately defined (usually binary) "dummy" variables, which play the role of the external *dimensions.* One widely used procedure for fitting a nonmetric version of this model is Kruskal's (1965, Kruskal & Carmone 1965) MONANOVA, and other nonmetric procedures for fitting this simple additive model include ADDIT (Roskam & Van Glist 1967), POLYCON (Young 1972), CM-I (Lingoes 1972, 1973), and ADDALS (de Leeuw et al 1976). ADDALS also allows more general cases in which, say, the factors of the factorial design are treated as ordinal or interval scale rather than (necessarily) nominal scale variables (or mixtures of scale types are allowed for factors), or in which the *dependent* variable is nominally scaled. Carroll's (1969) categorical conjoint measurement and Nishisato's (1971) optimal scaling approach also provide options for dealing with nominal scale dependent variables.

In recent years there have been increasingly frequent applications of conjoint measurement to data from experimental and other judgmental tasks (e.g. Cliff 1972, Ullrich & Painter 1974, Falmagne 1976), as well as relevant theoretical developments (e.g. Fishburn 1975, Falmagne et al 1979), which generally fall under the purview of a forthcoming chapter on unidimensional scaling and psychophysics in the *Annual Review of Psychology.* However, conjoint analysis remains one of the most underemployed techniques for data analysis in psychology. In contrast, the method has enjoyed extensive usage in marketing research, where Green & Wind (1973) provided a practitioner's handbook. Although applications are too numerous to cite at length, the following serve as examples: Johnson (1974), Green & Wind (1975), Green et al (1975), Bouroche (1977), Green (1977), Green & Carmone (1977). Helpful overviews of current developments in the application of conjoint analysis in marketing can be found in Green & Srinivasan (1978) and Wind (1976, 1978a,b).

A general procedure called ORDMET, that is applicable to nonmetric external analysis in terms of a scalar product model, is described by McClelland & Coombs (1975). Given data sufficiently close to being errorless, ORDMET's linear programming approach can be applied to fitting any external scalar product model, conjoint measurement model, and even external versions of the unfolding model.

Nonspatial Distance Models (for Two-mode Two-way Data)

The only nonspatial model proposed to date (outside the clustering literature) that is directly applicable to two-mode two-way data is Tversky & Sattath's (1979) "Preference Trees" model. This model is applicable to a paired comparisons preference matrix aggregated over subjects (or, more appropriately, over replications by a single subject) and so can be viewed (see above) as a two-mode two-way model. The Preference Tree (PRETREE) model follows as a special case of the elimination by aspects (EBA) model (Tversky 1972a,b) and subsumes Luce's (1959, 1977) constant ratio model. Although there is no program for fitting the PRETREE model to data, it has been tested by utilizing trees derived from similarity data or on a priori grounds.

TWO-MODE THREE-WAY DATA

As mentioned under one-mode two-way data, Carroll & Kruskal (1977) have provided a general overview of spatial models and data analytic methods falling under the present heading.

Spatial Distance Models (for Two-mode Three-way Data)

UNCONSTRAINED SYMMETRIC EUCLIDEAN MODELS (FOR TWO-MODE THREE-WAY DATA) The principal type of data falling under this classification is three-way dyadic data, comprising two or more square symmetric proximities matrices for pairs of stimuli, from two or more subjects (or other data sources). The dominant type of model is a distance model (only Euclidean models to date) for stimuli, with a set of individual differences parameters characterizing subjects. The models extend from the "points of view" approach of Tucker & Messick (1963) through various forms of a weighted Euclidean model (Bloxom 1968, Horan 1969, Carroll & Chang 1969, 1970), frequently called the INDSCAL (for INdividual Differences SCALing) model [but called a "subjective metrics" model by Schönemann (1972)].

Yet further generalizations of this model include Tucker's (1972) "three-mode scaling" model, Carroll & Chang's (1972a) IDIOSCAL model, and Harshman's (1972b) PARAFAC2 model. Of these IDIOSCAL is the most general, as it includes the other two as special cases. IDIOSCAL assumes a different *generalized* Euclidean metric, which for each subject is defined by a *quadratic form* described by a symmetric RxR matrix. Three-mode scaling is essentially a special case of Tucker's (1964) model for three-mode factor analysis, applied to an array of estimated scalar products derived from three-way proximities data. Tucker's approach can be viewed as a special case of the IDIOSCAL model, in which a special structure is imposed on the quadratic form matrices (that is, the individual quadratic forms are linear combinations of a small set of symmetric RxR matrices). Recent statistical developments in three-mode factor analysis are given by Bentler & Lee (1978, 1979). Harshman's (1972b) PARAFAC2 provides an interesting special case of both IDIOSCAL and three-way scaling. In terms of a geometric interpretation (also adopted by Tucker 1972), PARAFAC2 allows the dimensions to be oblique or correlated (i.e. have nonindependent effects on the data) but assumes that the angles between dimensions remain the same for all subjects.

All three of these models have the simple weighted Euclidean model INDSCAL (Carroll & Chang 1970) as a special case. INDSCAL has an important property, however, that two of these three more general models (IDIOSCAL and three-mode scaling) do not share, and which has only been conjectured but not proved for PARAFAC2 (Harshman 1972b). The specific feature is "dimensional uniqueness," which means that the dimensions are not invariant under orthogonal (or general linear) transformations, but are *uniquely* defined (or are "identifiable" in current statistical parlance) except for permutations and reflections. [See Harshman (1972a) and Kruskal (1976, 1977c) for uniqueness proofs. It should be noted that these results have actually been proved for the more general three-way CANDECOMP (for CANonical DECOMPosition) model provided by Carroll & Chang (1970) and independently by Harshman (1970) under the name of PARAFAC (for PARAllel FACtor analysis).] A more extensive discussion of these models can be found in Carroll (1973), Carroll & Wish (1974a,b) and Wish & Carroll (1974).

The principal algorithmic advances in this domain during recent years have entailed the procedure already discussed when considering the

ALSCAL method of Takane et al (1977) and Ramsay's (1977b) maximum likelihood approach in MULTISCALE. Each of these programs has both (one-mode) two-way and (two-mode) three-way capability. In the latter case, both techniques assume the weighted Euclidean, or INDSCAL model. The principal new feature of the ALSCAL treatment of three-way data is that the program provides a *nonmetric* implementation of the INDSCAL model. Another capability, also not available in other approaches (e.g. INDSCAL) to fitting the weighted Euclidean model, is the provision for missing or replicated data. In the case of MULTISCALE, which is restricted to the metric case, there are some points to be emphasized concerning three-way data. First, the asymptotic chi-square criterion for tests of statistical significance of dimensions is more questionable than in the two-way case. Ramsay (1979a) has devised an adjustment in degrees of freedom to expedite more valid nominal levels of significance. In addition, of course, MULTISCALE in the three-way case allows the definition of confidence regions for subject weights as well as for stimulus points. In research currently underway, Sharon Weinberg is comparing the confidence regions produced by MULTISCALE with those produced by straightforward jackknifing of INDSCAL, a less model-specific procedure employed earlier by Cohen (1974a) and Ebbesen (1977).

A mathematical development that has led to some important new algorithms for the INDSCAL model is Schönemann's (1972) "analytic solution for a class of subjective metrics models." This solution, however, is appropriate only for errorless data that fit the model exactly. More robust modifications have been provided by Carroll & Chang (1972a), Schönemann et al (1976), and de Leeuw & Pruzansky (1978). These three modifications all have the advantage that they provide approximate solutions for the weighted Euclidean or INDSCAL model in much less time than for the more standard implementations (Carroll & Chang 1970, Pruzansky 1975, Takane et al 1977, Ramsay 1977b). The solutions resulting from the more rapid algorithms often provide useful initial configurations for the standard approaches, which have more well-defined and probably more stable numerical properties. Another approach providing an initial configuration for the INDSCAL procedure is implemented in a program called SINDSCAL-LS (Carroll & Pruzansky 1979), based on a special case of CANDELINC (called LINCINDS) providing a linearly constrained version of INDSCAL. SINDSCAL-LS uses the stimulus space and/or subjects space from three-mode scaling to define the constraint matrices (cf Cohen 1974b, MacCallum 1976).

A final approach to be discussed here is one by Lingoes & Borg (1978), based generally on using "Procrustean" configuration matching techniques, called PINDIS (for Procrustean INdividual DIfferences Scaling). In addition to providing options for fitting models of the INDSCAL and IDIOSCAL variety, PINDIS introduces a new "vector weighting" or "perspective" model.

APPLICATIONS OF TWO-MODE THREE-WAY SYMMETRIC EUCLIDEAN MODELS Along with the increased capabilities of higher-way models, the user must accept the responsibility for offering convincing interpretations of a larger number of fitted parameters. Accordingly, the highly elegant work of Bisanz et al (1978) and LaPorte & Voss (1979) closely related the model parameters of INDSCAL solutions to substantive issues in the study of memory for prose. Other interesting results in the area of memory and cognition can be found in Shoben (1974), Howard & Howard (1977), and Friendly (1977). There have been many three-way analyses of perceptual data, including the studies reviewed in Carroll & Wish (1974a,b) and Wish & Carroll (1974). Other such papers include Carroll & Chang (1974), Walden & Montgomery (1975), Fraser (1976), Chang & Carroll (1978), Getty et al (1979), Soli & Arabie (1979), Arabie & Soli (1980). Researchers in social psychology and sociology have been especially active in applying the weighted Euclidean model (e.g. Rosenberg & Kim 1975; Breiger et al 1975; Wish 1975, 1976; Wish et al 1976; Wish & Kaplan 1977; Coxon & Jones 1978; Wish 1979a,b). The studies by Wish and his colleagues used the INDSCAL model and obtained substantive results supportive of Wish's implicit theory of interpersonal communication.

CONSTRAINED SYMMETRIC EUCLIDEAN MODELS (FOR TWO-MODE THREE-WAY DATA) A constrained approach to individual differences MDS that takes as its basic model the Tucker three-mode scaling model has been provided by Bloxom (1978), who imposes various equality constraints so that parameters are equal to each other or to prespecified values. Bloxom (1978) also includes a constrained version of the INDSCAL model as a special case, since INDSCAL itself corresponds to three-mode scaling with all (off diagonal) dimension cosines constrained to zero, and all three modes constrained to have the same number of dimensions.

A different approach to a constrained INDSCAL analysis is provided in a procedure called LINCINDS (for LINearly Constrained INDSCAL) that is a special case of the CANDELINC procedure (Carroll, Pruzansky & Kruskal 1979) to be discussed in detail under constrained three-mode three-way scalar product models. In LINCINDS the INDSCAL stimulus coordinates, subject weights, or both can be constrained to be linearly related to a set of exogenous ("outside") variables (measured on the stimuli, subjects, or both). Specifically, the coordinate x_{ir} of the ith stimulus on the rth dimension can be constrained to be of the form

$$x_{ir} = \sum_{s=1}^{S} b_{rs} v_{is} \quad ,$$

where v_{is} is the known value of stimulus i on exogenous variable s, and b_{rs} is a fitted coefficient (analogous to a regression coefficient in the least squares regression equation predicting dimension r from the S exogenous variables). In practice it has been found inappropriate to use this procedure to constrain the subject weights, however, both from empirical experience and for theoretical reasons related to MacCallum's (1977) criticism of applying linear procedures to INDSCAL weights, although we regard his arguments as overstated (see Carroll, Pruzansky & Kruskal 1979).

NONSYMMETRIC EUCLIDEAN MODELS (FOR TWO-MODE THREE-WAY DATA) DeSarbo (1978) has produced a three-way metric unfolding approach which can accommodate nonsymmetric data according to the first general principle listed above for such data. Also, as noted above in the discussion of Young's (1975a) ASYMSCAL, there is a three-way generalization of that model.

Scalar Product Models (for Two-mode Three-way Data)

While not originally formulated as such, both the INDSCAL (Carroll & Chang 1970) and three-mode scaling (Tucker 1972) procedures have been applied directly to scalar product data. Both methods ordinarily start out with dissimilarities data and, via preprocessing, transform these data into estimated scalar product matrices, which are then analyzed by a symmetric version of three-way CANDECOMP or of three-mode factor analysis, respectively. Either procedure just as easily fits a model directly for two-mode three-way scalar product data. Moreover, the INDSCAL program (Chang & Carroll 1969) and its successor SINDSCAL (Pruzansky 1975) both have options to deal with scalar product data directly.

A scalar product model explicitly formulated for nonsymmetric two-mode three-way data is a three-way version of Harshman's (1975, 1978) DEDICOM model. This is a generalization of the one-mode two-way DEDICOM model to the two-mode three-way case. A set of dimension weights analogous to those assumed in the INDSCAL-CANDECOMP-PARAFAC models are introduced as parameters describing the second mode (and third way), which may correspond to subjects or other data sources.

Another model for this type of nonsymmetric data has been formulated by Carroll & Sen (1976), and was explicitly designed for the case of "cross impact" data, in which each of a number of subjects judges the impact of each of a set of events on each other event. See Carroll (1977) for a description of the model and the corresponding analytic procedure, called Impact Scaling.

THREE-MODE THREE-WAY DATA

Spatial Distance Models (for Three-mode Three-way Data)

As already mentioned in the section on distance models for two-mode three-way nonsymmetric data, DeSarbo (1978) has implemented a three-way metric unfolding procedure, which can be interpreted either as a direct generalization of Schönemann's (1972) two-way metric unfolding model and method or as a nonsymmetric generalization of INDSCAL. The DeSarbo procedure, like Schönemann's, is both metric and unconditional (although a case can be made that DeSarbo's approach is *matrix* conditional). A typical data array to which this three-way unfolding procedure can be applied is a set of subjects by stimuli matrices of preference scale values, one such matrix for each of a number of situations or experimental conditions.

Scalar Product Models (for Three-mode Three-way Data)

The two principal unconstrained models appropriate to this section are the Tucker (1964) three-mode factor analysis model and the general three-way case of CANDECOMP-PARAFAC. While there have been some useful applications of three-mode factor analysis (Wiggins & Blackburn 1976, Sjoberg 1977, Redfield & Stone 1979), there have so far been very few convincing applications of the general three-mode three-way case of CANDECOMP-PARAFAC (but see Harshman, Ladefoged & Goldstein 1977). CANDECOMP has mainly been useful (in its two-mode three-way symmetric case) as the analytic underpinnings of the Carroll-Chang INDSCAL procedure and, more recently, as the first step in DeSarbo's (1978) approach to three-way unfolding.

Turning now to constrained models, we note that the CANDELINC approach, which has been referred to previously (Carroll, Pruzansky & Kruskal 1979), is directly applicable to the three- or higher-way CANDECOMP model. In the three-way case, CANDELINC allows linear constraints on *all* modes, or on just one or two of the three modes. In general these constraints take the form that the parameters for a given mode must be linear combinations of a set of a priori external variables. These external variables are defined via a "design matrix" for each of the linearly constrained modes, with the design matrix containing the evaluated external variables.

HIGHER-WAY DATA

Tucker's three-mode factor analysis could easily be generalized to the higher-way case (see Carroll & Wish 1974a,b), but to our knowledge no actual implementation has been achieved. The N-way CANDECOMP model has been so implemented. While it has not been usefully applied to general N-way multivariate data, one particular useful application has been to a least squares fitting of the Lazarsfeld latent class model (Carroll, Pruzansky & Green 1979).

DATA COLLECTION AND RELATED ISSUES

Although we have emphasized the development of models and their algorithms, there has also been much research in the techniques used before the model is to be fitted, including the perennial problem of comparing two or more proximities matrices. The fact that the straightforward approaches (e.g. correlating two matrices) encounter formidable difficulties when inferential tests are sought has often caused investigators to feel that only the scaling output (but not the input matrices) could be compared. The consequent practices have recently become less forgivable owing to results (Hubert & Schultz 1976b, Hubert 1978) which generalized earlier work of Mantel (1967) to allow significance tests for the correspondence between two or more (Hubert 1979) input matrices as well as related applications.

The extensive variety of models (and their associated types of input data) notwithstanding, situations often arise where the data at hand are not immediately compatible with the intended model. A typical example occurs when a one-mode two-way nonmetric scaling representation is sought for either of the modes of a two-mode two-way data set. Shepard (1972b) has labeled as "indirect similarities" (also sometimes called "profile similarities") the secondary data that ultimately serve as input to the program implementing the model. An example consists of computing the squared Euclidean distances (cf Carroll 1968) between all pairs of rows/columns of such a two-mode two-way data set.

One relevant area of research concerns the partitions that result when subjects are asked to sort a set of stimuli into "homogeneous groups." For analyses where differences between *subjects'* sortings are of interest, a variety of measures of distance between such partitions have been developed (Boorman & Arabie 1972, Arabie & Boorman 1973). For situations in which the *stimuli* being sorted are of greater interest in the analysis, there is an extensive literature on techniques for going from partitions of the stimuli to one-mode two-way (stimuli by stimuli) data: Carroll (1968), Rosenberg & Jones (1972), Rosenberg & Sedlak (1972), Rosenberg & Kim

(1975), Wish (1976), Wish et al (1976), Wish & Kaplan (1977), Drasgow & Jones (1979). Other papers relevant to indirect similarities data include Sibson (1972), Lund (1974), Batchelder & Narens (1977), Arabie & Soli (1980). Finally, R. A. Harshman (personal communication) has reported favorable results when two-way marginals are subtracted from three-way data in applications of CANDECOMP-PARAFAC, and Kruskal (1977d) has derived least squares properties supporting this strategy.

A related area of activity in scaling concerns the development of incomplete experimental designs to reduce the effort and expense involved in collecting MDS data. For selectively obtaining data on a subset of the $n(n-1)/2$ pairs of n stimuli, the following may serve as useful references: Spence & Domoney (1974), Green & Bentler (1979), Deutscher (1980), Green (1980), Isaac (1980), Kohler & Rushton (1980), Spence (1980), Young et al (1980). For conjoint analysis, Green et al (1978) have discussed an approach that spares researchers the need to execute a full factorial design.

MDS: NEW AREAS OF USAGE

In addition to research activities in the United States, Canada, the United Kingdom, the Netherlands, Israel, and Sweden, various other countries have developed their own traditions of MDS. In Japan, Hayashi, Indow, and others have been especially active (see references throughout this chapter), and Okada and Watanabe have translated into Japanese the two volumes of the 1969 Irvine conference (Shepard et al 1976, Romney et al 1977). Bouroche and his colleagues in France have been responsible for many developments and applications of scaling techniques (Bertier & Bouroche 1975, Bouroche & Dussaix 1975). In Germany, Feger (1978) and Bick and Müller have formed the core of groups actively developing and using MDS and related methods. In the Soviet Union, there is continuing work by Mirkin and others (Terekhin 1973, 1974; Kamenskii 1977).

With respect to disciplines, MDS has maintained a strong base in marketing research. There also appear to be possible applications in econometrics (Maital 1978), and usage in political science (e.g. Weisberg 1972) and sociology (e.g. Boorman & White 1976; Coxon & Jones 1977) is also apparent from various references cited earlier. From an advocacy point of view, perhaps the greatest gains in areas related to psychology have come from geography (e.g. Tobler & Wineburg 1971, Olshavsky et al 1975, Golledge & Spector 1978, Golledge et al 1980; also see other papers in Golledge & Rayner 1980). We view this substantive interest in MDS from related disciplines as providing a salutary diversity of assumptions upon which new models can be formulated.

AIDS TO USERS: TEXTBOOKS

MDS remains an area characterized by a considerable lag between new methodological developments and routine use by nonspecialists (viz., the majority of the consumer community). The fact that the two-volume Irvine conference proceedings (Shepard et al 1972, Romney et al 1972) were never intended to be a textbook has frustrated many instructors, and so have the ongoing developments subsequent to publication of some of the most useful textbooks (Dawes 1972, Green & Rao 1972, Green & Wind 1973). Fortunately, a monograph by Kruskal & Wish (1978) has recently appeared, and it is eminently usable as a textbook covering two- and three-way MDS of proximities data. This monograph provides helpful guidelines to and examples of usage, and has been enthusiastically received by graduate and advanced undergraduate students in courses we have taught.

PROSPECTS

As stated in our introduction, our primary goal in this chapter has been to impose a taxonomy on current models and methods so that their interrelationships as well as various lacunae would be more apparent. While reviewing developments and applications in MDS, we have noted several trends. First, there is increased attention to the substantive appropriateness of these models in contrast to earlier years when the techniques served primarily as convenient vehicles (and sometimes steam rollers) for data reduction. Second, we find increased realization that no particular model, in general, gives "the true representation." Most analyses choose a model that at best captures part of the structure inherent in the data; the part not fitted often awaits another analysis with a different model and perhaps a complementary interpretation as well. Third, we see a strong trend toward the development of three-way models with applications of three- and higher-way methods becoming almost as numerous as two-way applications. A development not unrelated to the two preceding observations is that we see considerable interest in discrete and hybrid models and predict that their coverage will be more extensive in the next chapter on MDS in this series.

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CONTENTS